Polynomials: Factorization, H.C.F. & L.C.M., Rational Functions, and Polynomial Division

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Factorization of Polynomials

- Extracting common factors
- Grouping terms
- Using identities (difference of squares, perfect square trinomials)
- Cross method for quadratics

Example 1.1: Factorize $x^2 + 5x + 6$

Step 1: Identify coefficients.
We have a quadratic $ax^2 + bx + c$ with $a = 1$, $b = 5$, $c = 6$.

Step 2: Find two numbers that multiply to $a \cdot c = 1 \cdot 6 = 6$ and add to $b = 5$.
These numbers are 2 and 3 because $2 \times 3 = 6$ and $2 + 3 = 5$.

Step 3: Rewrite the middle term $5x$ using $2x + 3x$.

$$x^2 + 5x + 6 = x^2 + 2x + 3x + 6.$$

Step 4: Group into two pairs and factor each pair.

$$= \left(x^2 + 2x\right) + \left(3x + 6\right) = x(x + 2) + 3(x + 2).$$

Step 5: Factor out the common binomial factor $(x + 2)$.

$$= (x + 2)(x + 3).$$

Thus, $x^2 + 5x + 6 = (x + 2)(x + 3)$.

Example 1.2: Factorize $2x^2 + 7x + 3$

Step 1: Identify $a = 2$, $b = 7$, $c = 3$.
Compute $a \cdot c = 2 \times 3 = 6$.

Step 2: Find two numbers that multiply to 6 and add to 7.
Numbers are 6 and 1 because $6 \times 1 = 6$ and $6 + 1 = 7$.

Step 3: Split the middle term $7x$ as $6x + x$.

$$2x^2 + 7x + 3 = 2x^2 + 6x + x + 3.$$

Step 4: Group and factor each pair.

$$= (2x^2 + 6x) + (x + 3) = 2x(x + 3) + 1(x + 3).$$

Step 5: Factor out $(x + 3)$.

$$= (x + 3)(2x + 1).$$

Therefore, $2x^2 + 7x + 3 = (x + 3)(2x + 1)$.

Example 1.3: Factorize $x^2 - 9$

Recognize a difference of squares pattern.
A difference of squares has form $A^2 - B^2 = (A - B)(A + B)$.

Step 1: Identify $A = x$, $B = 3$ since $x^2 - 9 = x^2 - 3^2$.

Step 2: Apply the identity.

$$x^2 - 9 = (x - 3)(x + 3).$$

That completes the factorization.

Example 1.4: Factorize $x^3 - x^2 - x + 1$

Step 1: Group terms to factor by grouping.

$$x^3 - x^2 - x + 1 = (x^3 - x^2) + (-x + 1).$$

Step 2: Factor out the greatest common factor (GCF) from each group.

$$= x^2(x - 1) - 1(x - 1).$$

Step 3: Notice $(x - 1)$ is common.

$$= (x^2 - 1)(x - 1).$$

Step 4: Recognize $x^2 - 1$ is a difference of squares.

$$x^2 - 1 = (x - 1)(x + 1).$$

Step 5: Combine all factors.

$$x^3 - x^2 - x + 1 = (x - 1)(x - 1)(x + 1) = (x - 1)^2(x + 1).$$

Example 1.5: Factorize $3x^2 - 12$

Step 1: Identify a common numerical factor.
Each term, $3x^2$ and -12, is divisible by 3.

Step 2: Factor out 3.

$$3x^2 - 12 = 3(x^2 - 4).$$

Step 3: Factor inside parentheses as difference of squares.

$$x^2 - 4 = (x - 2)(x + 2).$$

Step 4: Combine factors.

$$3x^2 - 12 = 3(x - 2)(x + 2).$$

That completes the factorization.

Example 1.6: Factorize $x^4 - 5x^2 + 4$

Step 1: Let $y = x^2$. Rewrite as quadratic in y.

$$x^4 - 5x^2 + 4 = y^2 - 5y + 4 \quad (\text{where } y = x^2).$$

Step 2: Factor $y^2 - 5y + 4$.

Find numbers multiplying to 4 and summing to -5: -1 and -4.

$$y^2 - 5y + 4 = (y - 1)(y - 4).$$

Step 3: Substitute back $y = x^2$.

$$= (x^2 - 1)(x^2 - 4).$$

Step 4: Factor each as difference of squares.

$$x^2 - 1 = (x - 1)(x + 1), \quad x^2 - 4 = (x - 2)(x + 2).$$

Step 5: Combine all factors.

$$x^4 - 5x^2 + 4 = (x - 1)(x + 1)(x - 2)(x + 2).$$

Practice: Factorization (1 of 3)

1. **Factorize** $x^2 + 6x + 8$.
2. **Factorize** $3x^2 - x - 4$.
3. **Factorize** $4x^2 - 9$.
4. **Factorize** $x^3 + x^2 - x - 1$.
5. **Factorize** $2x^3 - 8x$.

Practice: Factorization (2 of 3)

(6) **Factorize** $x^4 - 16$.

(7) **Factorize** $x^3 - 6x^2 + 11x - 6$.

(8) **Factorize** $5x^2 + 20x + 15$.

(9) **Factorize** $x^3 + 3x^2 - 4x - 12$.

(10) **Factorize** $6x^2 - x - 2$.

(11) **Factorize** $x^4 - 10x^2 + 9$.

(12) **Factorize** $2x^3 + 5x^2 - 2x - 5$.

(13) **Factorize** $x^3 - 4x$.

(14) **Factorize** $3x^2 + 2x - 8$.

(15) **Factorize** $x^4 - 1$.

Solutions: Factorization

1. $(x+2)(x+4)$.
2. $(3x+4)(x-1)$.
3. $(2x-3)(2x+3)$.
4. $(x+1)^2(x-1)$.
5. $2x(x-2)(x+2)$.
6. $(x-2)(x+2)(x^2+4)$.
7. $(x-1)(x-2)(x-3)$.
8. $5(x+1)(x+3)$.
9. $(x+3)(x-2)(x+2)$.
10. $(3x-4)(x+2)$.
11. $(x-1)(x+1)(x-3)(x+3)$.
12. $(x-1)(x+1)(2x+5)$.
13. $x(x-2)(x+2)$.
14. $(3x-4)(x+2)$.
15. $(x-1)(x+1)(x^2+1)$.

H.C.F. and L.C.M. of Polynomials

- **H.C.F.**: Highest-degree polynomial dividing all given polynomials.
- **L.C.M.**: Lowest-degree polynomial that is a multiple of all given polynomials.

Example 2.1: H.C.F. & L.C.M. of $x^2y^3z^4$ and $x^2y^5z^2$

Step 1: Express each in prime factor form.

$$x^2y^3z^4 = x^2 \cdot y^3 \cdot z^4, \quad x^2y^5z^2 = x^2 \cdot y^5 \cdot z^2.$$

Step 2: For H.C.F., take the lowest power of each variable common to both.

$$x : \min(2,2) = 2, \quad y : \min(3,5) = 3, \quad z : \min(4,2) = 2.$$
$$\text{H.C.F.} = x^2y^3z^2.$$

Step 3: For L.C.M., take the highest power of each variable present in any factor.

$$x : \max(2,2) = 2, \quad y : \max(3,5) = 5, \quad z : \max(4,2) = 4.$$
$$\text{L.C.M.} = x^2y^5z^4.$$

Example 2.2: H.C.F. & L.C.M. of $2(x-1)(x+2)$ and $3(x-1)^2(x+3)$

Step 1: Write factorized forms with numerical coefficients included.

$$2(x-1)(x+2), \quad 3(x-1)^2(x+3).$$

Step 2: Identify common polynomial factors.
- Both have at least one factor of $(x-1)$.
- $(x+2)$ appears only in the first, $(x+3)$ appears only in the second.

Step 3: H.C.F. takes common polynomial factors with lowest power.
Common: $(x-1)^1$. Numerical: $\gcd(2,3)=1$.

$$\text{H.C.F.} = (x-1).$$

Step 4: L.C.M. takes each distinct factor with highest exponent and product of numerical LCM.
Numerical L.C.M. of 2 and 3 is 6.
Include $(x-1)^2$ (highest power from second), $(x+2)$, and $(x+3)$.

Example 2.3: H.C.F. & L.C.M. of $2x^2y - 2xy^2$ and $4xy^3 - 4x^2y$

Step 1: Factor each expression completely.

$$2x^2y - 2xy^2 = 2xy(x - y),$$
$$4xy^3 - 4x^2y = 4xy(y^2 - x) = -4xy(x - y^2).$$

Since $y^2 - x = -(x - y^2)$, but to compare $(x - y)$ vs $(x - y^2)$, not common polynomial. We'll factor second differently:

$$4xy^3 - 4x^2y = 4xy(y^2 - x) = 4xy\left(-1\right)\left(x - y^2\right) = -4xy(x - y^2).$$

Step 2: Identify common factors.
Both have numerical factor and xy. Numerical: $\gcd(2, 4) = 2$. Polynomial: common factor xy. The binomial factors $(x - y)$ and $(x - y^2)$ are different.

Thus,

$$\text{H.C.F.} = 2xy.$$

Step 3: Find L.C.M. as product of each factor to highest power,

Example 2.4: H.C.F. & L.C.M. of $x^2 - 1$ and $x^2 - x$

Step 1: Factor each polynomial.

$$x^2 - 1 = (x-1)(x+1),$$
$$x^2 - x = x(x-1).$$

Step 2: Identify common factors.
Common factor is $(x-1)$. Numerical parts are both 1.

$$\text{H.C.F.} = (x-1).$$

Step 3: Combine factors for L.C.M.
Include each distinct factor the greatest number of times it appears: - $(x-1)$ appears to first power in both. - Also include $(x+1)$ from the first and x from the second.

$$\text{L.C.M.} = x(x-1)(x+1).$$

Example 2.5: H.C.F. & L.C.M. of $x^3 - x$ and $x^2 - 1$

Step 1: Factor each.

$$x^3 - x = x(x^2 - 1) = x(x-1)(x+1),$$
$$x^2 - 1 = (x-1)(x+1).$$

Step 2: Common factors: each has $(x-1)(x+1)$.

$$\text{H.C.F.} = (x-1)(x+1).$$

Step 3: L.C.M. includes all factors at highest power:
- From first: $x(x-1)(x+1)$, from second: $(x-1)(x+1)$. Combine gives $x(x-1)(x+1)$.

$$\text{L.C.M.} = x(x-1)(x+1).$$

Example 2.6: H.C.F. & L.C.M. of $3x^2y - 6xy^2$ and $9x^3y^2 - 3xy^3$

Step 1: Factor each.

$$3x^2y - 6xy^2 = 3xy(x - 2y),$$
$$9x^3y^2 - 3xy^3 = 3xy(3x^2y - y^2) = 3xy \cdot y\,(3x^2 - y).$$

Thus $9x^3y^2 - 3xy^3 = 3xy^2(3x^2 - y)$.

Step 2: Common factors:
Numerical: $\gcd(3, 9) = 3$. Polynomial: both have xy.

$$\text{H.C.F.} = 3xy.$$

Step 3: L.C.M. uses highest powers:
Numerical LCM of 3 and 9 is 9. The polynomial factors xy, $(x - 2y)$, and $y(3x^2 - y) = y(3x^2 - y)$.

Practice: H.C.F. & L.C.M. (1 of 3)

1. **Find H.C.F. and L.C.M. of** $3x^2y^2$ and $6xy^3$.
2. **Find H.C.F. and L.C.M. of** $4x^3 - 4x$ and $2x^2 - 2$.
3. **Find H.C.F. and L.C.M. of** $x^2 - 4x$ and $x^3 - 4x^2$.
4. **Find H.C.F. and L.C.M. of** $(x-2)(x+3)$ and $(x-2)^2(x-1)$.
5. **Find H.C.F. and L.C.M. of** $2x^2y - 4xy^2$ and $6x^3y^2 - 3xy^3$.

(6) **Find H.C.F. and L.C.M. of** $5x^2 - 20$ and $3x^2 - 12$.

(7) **Find H.C.F. and L.C.M. of** $x^3 - 8$ and $x^2 - 4$.

(8) **Find H.C.F. and L.C.M. of** $x^2y - xy^2$ and $xy^3 - x^2y^2$.

(9) **Find H.C.F. and L.C.M. of** $x^4 - 1$ and $x^2 - 1$.

(10) **Find H.C.F. and L.C.M. of** $2x^3 - 6x$ and $4x^2 - 12$.

Practice: H.C.F. & L.C.M. (3 of 3)

(11) **Find H.C.F. and L.C.M. of** $3x^2y - 9xy$ and $6x^3y^2 - 3xy^2$.

(12) **Find H.C.F. and L.C.M. of** $x^3 + x^2 - x - 1$ and $x^2 - 1$.

(13) **Find H.C.F. and L.C.M. of** $4x^2 - 9x + 5$ and $2x^2 - 3x + 1$.

(14) **Find H.C.F. and L.C.M. of** $x^3 - 27$ and $x^2 - 9$.

(15) **Find H.C.F. and L.C.M. of** $x^2 - x - 6$ and $x^3 - x^2 - 6x$.

Solutions: H.C.F. & L.C.M.

1. H.C.F. $= 3xy^2$, L.C.M. $= 6x^2y^3$.
2. H.C.F. $= 2(x^2 - 1)$, L.C.M. $= 4x(x^2 - 1)$.
3. H.C.F. $= x(x - 4)$, L.C.M. $= x^2(x - 4)$.
4. H.C.F. $= (x - 2)$, L.C.M. $= (x - 2)^2(x - 1)(x + 3)$.
5. H.C.F. $= xy$, L.C.M. $= 6xy^2(x - 2y)(2x^2 - y)$.
6. H.C.F. $= (x^2 - 4)$, L.C.M. $= 15(x^2 - 4)$.
7. H.C.F. $= (x - 2)$, L.C.M. $= (x - 2)(x + 2)(x^2 + 2x + 4)$.
8. H.C.F. $= xy(x - y)$, L.C.M. $= xy^2(x - y)$.
9. H.C.F. $= (x^2 - 1)$, L.C.M. $= (x^2 - 1)(x^2 + 1)$.
10. H.C.F. $= 2(x^2 - 3)$, L.C.M. $= 4x(x^2 - 3)$.
11. H.C.F. $= 3xy$, L.C.M. $= 3xy^2(x - 3)(2x^2 - 1)$.
12. H.C.F. $= (x - 1)(x + 1)$, L.C.M. $= (x - 1)^2(x + 1)$.
13. H.C.F. $= (x - 1)$, L.C.M. $= (4x - 5)(2x - 1)(x - 1)$.
14. H.C.F. $= (x - 3)$, L.C.M. $= (x - 3)(x + 3)(x^2 + 3x + 9)$.
15. H.C.F. $= (x - 3)(x + 2)$, L.C.M. $= x(x - 3)(x + 2)$.

Simplification of Rational Functions

To simplify $\frac{P(x)}{Q(x)}$:

1. Factorize numerator $P(x)$ and denominator $Q(x)$.
2. Cancel any common factors, ensuring excluded values where denominator is zero.

Example 3.1: Simplify $\dfrac{x^2 - 9}{x^2 - 6x + 9}$

Step 1: Factor numerator and denominator.

$$x^2 - 9 = (x-3)(x+3), \quad x^2 - 6x + 9 = (x-3)^2.$$

Step 2: Write fraction with factors.

$$\frac{(x-3)(x+3)}{(x-3)(x-3)}.$$

Step 3: Cancel one $(x-3)$ factor from numerator and denominator.

$$= \frac{x+3}{x-3}, \quad x \neq 3.$$

Result: $\frac{x^2-9}{x^2-6x+9} = \frac{x+3}{x-3}$, with $x \neq 3$.

Example 3.2: Simplify $\dfrac{2x^2 - 8}{4x}$

Step 1: Factor numerator.

$$2x^2 - 8 = 2(x^2 - 4) = 2(x - 2)(x + 2).$$

Step 2: Write denominator as $4x$.

$$\frac{2(x-2)(x+2)}{4x}.$$

Step 3: Simplify numerical factor: $\frac{2}{4} = \frac{1}{2}$.

$$= \frac{(x-2)(x+2)}{2x}, \quad x \neq 0.$$

Example 3.3: Simplify $\dfrac{3x^2 + 6x}{3x}$

Step 1: Factor numerator completely.

$$3x^2 + 6x = 3x(x + 2).$$

Step 2: Write fraction.

$$\frac{3x(x + 2)}{3x}.$$

Step 3: Cancel the common factor $3x$.

$$= x + 2, \quad x \neq 0.$$

Example 3.4: Simplify $\dfrac{x^3 - x}{x^2 - 1}$

Step 1: Factor numerator and denominator.

$$x^3 - x = x(x^2 - 1) = x(x-1)(x+1),$$
$$x^2 - 1 = (x-1)(x+1).$$

Step 2: Write fraction.

$$\frac{x(x-1)(x+1)}{(x-1)(x+1)}.$$

Step 3: Cancel $(x-1)(x+1)$.

$$= x, \quad x \neq \pm 1.$$

Example 3.5: Simplify $\dfrac{x^2 - 4x + 4}{x^2 - 4}$

Step 1: Factor numerator and denominator.

$$x^2 - 4x + 4 = (x-2)^2, \quad x^2 - 4 = (x-2)(x+2).$$

Step 2: Write fraction.

$$\frac{(x-2)(x-2)}{(x-2)(x+2)}.$$

Step 3: Cancel one $(x-2)$.

$$= \frac{x-2}{x+2}, \quad x \neq \pm 2.$$

Example 3.6: Simplify $\dfrac{4x^3 - 8x^2}{2x^2 - 4x}$

Step 1: Factor numerator and denominator.

$$4x^3 - 8x^2 = 4x^2(x-2), \quad 2x^2 - 4x = 2x(x-2).$$

Step 2: Write fraction.

$$\frac{4x^2(x-2)}{2x(x-2)}.$$

Step 3: Cancel $(x-2)$ and simplify numerical factor $\frac{4x^2}{2x} = 2x$.

$$= 2x, \quad x \neq 0, 2.$$

Practice: Simplification (1 of 3)

1. **Simplify** $\dfrac{x^2 - 5x + 6}{x^2 - 4}$.
2. **Simplify** $\dfrac{4x^2 - 9}{2x^2 + 3x}$.
3. **Simplify** $\dfrac{x^3 - 4x}{x^2 - x}$.
4. **Simplify** $\dfrac{5x^2 + 10x}{15x}$.
5. **Simplify** $\dfrac{2x^2 - 8x + 8}{x^2 - 2x}$.

Practice: Simplification (2 of 3)

(6) **Simplify** $\dfrac{x^2 - x - 6}{x^2 - 9}$.

(7) **Simplify** $\dfrac{3x^2 - 12}{6x}$.

(8) **Simplify** $\dfrac{x^3 + 2x^2 - x - 2}{x^2 - 1}$.

(9) **Simplify** $\dfrac{4x^2 - 6x}{2x - 3}$.

(10) **Simplify** $\dfrac{x^2 - 16}{x^2 - 8x + 16}$.

Practice: Simplification (3 of 3)

(11) **Simplify** $\dfrac{x^3 - x^2 - 6x}{x - 3}$.

(12) **Simplify** $\dfrac{6x^2 - 9x}{3x}$.

(13) **Simplify** $\dfrac{x^4 - x^2}{x^2 - 1}$.

(14) **Simplify** $\dfrac{2x^3 - 8x}{4x^2 - 16}$.

(15) **Simplify** $\dfrac{3x^3 + 6x^2 - 9x}{3x}$.

Solutions: Simplification

1. $\dfrac{x-3}{x+2}, \quad x \neq \pm 2.$
2. $\dfrac{2x-3}{x}, \quad x \neq 0, -\dfrac{3}{2}.$
3. $\dfrac{(x-2)(x+2)}{x-1}, \quad x \neq 0, 1.$
4. $\dfrac{x+2}{3}, \quad x \neq 0.$
5. $\dfrac{2(x-2)}{x}, \quad x \neq 0, 2.$
6. $\dfrac{x+2}{x+3}, \quad x \neq \pm 3.$
7. $\dfrac{(x-2)(x+2)}{2x}, \quad x \neq 0.$
8. $x+2, \quad x \neq \pm 1.$
9. $2x, \quad x \neq \frac{3}{2}.$
10. $\dfrac{x+4}{x-4}, \quad x \neq 4.$
11. $x^2+2x+3, \quad x \neq 3.$

Worked Examples (1 of 2)

Example 4.1

$$\frac{x^2-9}{x^2-4} \times \frac{x^2-1}{x^2-6x+9} = \frac{(x-3)(x+3)}{(x-2)(x+2)} \times \frac{(x-1)(x+1)}{(x-3)^2}.$$

Cancel one $(x-3)$ factor:

$$= \frac{(x+3)(x-1)(x+1)}{(x-2)(x+2)(x-3)}.$$

Example 4.2

$$\frac{2x^2-8}{x^2-1} \div \frac{x^2-4}{2x} = \frac{2(x-2)(x+2)}{(x-1)(x+1)} \times \frac{2x}{(x-2)(x+2)} = \frac{4x}{(x-1)(x+1)}.$$

Worked Examples (2 of 2)

Example 4.3

$$\frac{3x}{x^2-9} \times \frac{x^2-x-6}{x} = \frac{3x}{(x-3)(x+3)} \times \frac{(x-3)(x+2)}{x}.$$

Cancel x and $(x-3)$:

$$= \frac{3(x+2)}{x+3}.$$

Example 4.4

$$\frac{x^2-4}{3x} \div \frac{x^2-1}{6} = \frac{(x-2)(x+2)}{3x} \times \frac{6}{(x-1)(x+1)} = \frac{2(x-2)(x+2)}{x(x-1)(x+1)}.$$

Example 4.5

$$\frac{x^3-x}{x^2-1} \div \frac{2x^2-2x}{x} = \frac{x(x-1)(x+1)}{(x-1)(x+1)} \times \frac{x}{2x(x-1)} = \frac{x}{2(x+1)(x-1)}.$$

Practice: Multiply/Divide (1 of 3)

1. **Compute** $\dfrac{x^2-9}{x^2-4} \times \dfrac{x^2-1}{x^2-6x+9}$.
2. **Compute** $\dfrac{2x^2-8}{x^2-1} \div \dfrac{x^2-4}{2x}$.
3. **Compute** $\dfrac{3x}{x^2-9} \times \dfrac{x^2-x-6}{x}$.
4. **Compute** $\dfrac{x^2-4}{3x} \div \dfrac{x^2-1}{6}$.
5. **Compute** $\dfrac{x^3-x}{x^2-1} \div \dfrac{2x^2-2x}{x}$.

Practice: Multiply/Divide (2 of 3)

(6) **Compute** $\dfrac{2x^2 - x - 1}{x + 2} \times \dfrac{x^2 - 4}{x^2 - x - 2}$.

(7) **Compute** $\dfrac{4x^2 - 9}{2x^2 + 3x} \times \dfrac{x^2 - x - 6}{x^2 - 9}$.

(8) **Compute** $\dfrac{x^2 - 16}{x^2 - 8x + 16} \div \dfrac{4x^2 - 6x}{2x - 3}$.

(9) **Compute** $\dfrac{3x^2 + 6x - 9}{3x} \times \dfrac{x^2 - 1}{x + 1}$.

(10) **Compute** $\dfrac{x^3 - 4x}{x^2 - x} \div \dfrac{5x^2 + 10x}{15x}$.

Practice: Multiply/Divide (3 of 3)

(11) **Compute** $\dfrac{2x^3 - 8x}{4x^2 - 16} \times \dfrac{3x^2 - 9x}{3x}$.

(12) **Compute** $\dfrac{x^4 - x^2}{x^2 - 1} \div \dfrac{2x^3 - 8x}{4x^2 - 16}$.

(13) **Compute** $\dfrac{5x^2 + 15x}{10x} \times \dfrac{x^2 - 6x + 9}{x - 3}$.

(14) **Compute** $\dfrac{4x^3 - 8x^2}{2x^2 - 4x} \div \dfrac{x^2 - 4}{3x}$.

(15) **Compute** $\dfrac{3x^2 - x - 2}{x + 1} \times \dfrac{x^2 - 9}{x^2 - 6x + 9}$.

Solutions: Multiply/Divide

1. $\dfrac{(x-3)(x-1)(x+1)}{(x-2)(x+2)(x-3)}$.
2. $\dfrac{4x}{(x-1)(x+1)}$.
3. $\dfrac{3(x+2)}{x+3}$.
4. $\dfrac{2(x-2)(x+2)}{x(x-1)(x+1)}$.
5. $\dfrac{x}{2(x-1)(x+1)}$.
6. $\dfrac{(2x+1)(x-1)}{x+1}$.
7. $\dfrac{(2x-3)(x+2)}{x(x+3)}$.
8. $\dfrac{x+4}{x(x-4)}$.
9. $\dfrac{(x-1)^2(x+2)}{x}$.

Worked Examples (1 of 2)

Example 5.1

$$\frac{1}{x-2} + \frac{1}{x+2} = \frac{(x+2)+(x-2)}{(x-2)(x+2)} = \frac{2x}{x^2-4}.$$

Example 5.2

$$\frac{x}{x-1} - \frac{2}{x+1} = \frac{x(x+1)-2(x-1)}{(x-1)(x+1)} = \frac{x^2+x-2x+2}{x^2-1} = \frac{x^2-x+2}{x^2-1}.$$

Example 5.3

$$\frac{2x}{x^2-9}+\frac{3}{x-3}=\frac{2x}{(x-3)(x+3)}+\frac{3(x+3)}{(x-3)(x+3)}=\frac{2x+3x+9}{x^2-9}=\frac{5x+9}{x^2-9}.$$

Example 5.4

$$\frac{x^2}{x^2-4}-\frac{x}{x-2}=\frac{x^2}{(x-2)(x+2)}-\frac{x(x+2)}{(x-2)(x+2)}=\frac{x^2-x^2-2x}{x^2-4}=\frac{-2x}{x^2-4}$$

Example 5.5

$$\frac{3x}{x^2-1}+\frac{2}{x+1}=\frac{3x}{(x-1)(x+1)}+\frac{2(x-1)}{(x+1)(x-1)}=\frac{3x+2x-2}{x^2-1}=\frac{5x-2}{x^2-1}.$$

Example 5.6

Practice: Add/Subtract (1 of 3)

1. **Compute** $\dfrac{1}{x-2} + \dfrac{1}{x+2}$.
2. **Compute** $\dfrac{x}{x-1} - \dfrac{2}{x+1}$.
3. **Compute** $\dfrac{2x}{x^2-9} + \dfrac{3}{x-3}$.
4. **Compute** $\dfrac{x^2}{x^2-4} - \dfrac{x}{x-2}$.
5. **Compute** $\dfrac{3x}{x^2-1} + \dfrac{2}{x+1}$.

Practice: Add/Subtract (2 of 3)

(6) **Compute** $\dfrac{x-1}{x+2} - \dfrac{x+1}{x-2}$.

(7) **Compute** $\dfrac{2x}{x^2-1} - \dfrac{1}{x-1}$.

(8) **Compute** $\dfrac{x+2}{x^2-4} + \dfrac{x-2}{x+2}$.

(9) **Compute** $\dfrac{x^2-1}{x^2-9} - \dfrac{x}{x-3}$.

(10) **Compute** $\dfrac{3x+1}{x^2-4} + \dfrac{2x-2}{x-2}$.

Practice: Add/Subtract (3 of 3)

(11) **Compute** $\dfrac{x^2 - 4x + 3}{x - 1} - \dfrac{x}{x + 1}$.

(12) **Compute** $\dfrac{2x^2 - 2}{x^2 - 1} + \dfrac{x + 1}{x - 1}$.

(13) **Compute** $\dfrac{4x}{x^2 - 9} - \dfrac{2}{x + 3}$.

(14) **Compute** $\dfrac{5x + 2}{x^2 - 4} + \dfrac{x - 2}{x - 2}$.

(15) **Compute** $\dfrac{x^2 - x - 6}{x^2 - 4} - \dfrac{x}{x + 2}$.

Solutions: Add/Subtract

1. $\dfrac{2x}{x^2 - 4}$.
2. $\dfrac{x^2 - x + 2}{x^2 - 1}$.
3. $\dfrac{5x + 9}{x^2 - 9}$.
4. $\dfrac{-2x}{x^2 - 4}$.
5. $\dfrac{5x - 2}{x^2 - 1}$.
6. $\dfrac{-6x}{x^2 - 4}$.
7. $\dfrac{x + 1}{x^2 - 1}$.
8. $\dfrac{x^2 - 3x + 6}{x^2 - 4}$.
9. $\dfrac{-3x - 1}{x^2 - 9}$.
10. $2x^2 + 3x - 3$

Long Division Method

Procedure:

1. Arrange both dividend and divisor in descending powers of x.
2. Divide the leading term of dividend by leading term of divisor to find first term of quotient.
3. Multiply divisor by this term and subtract from dividend.
4. Repeat with new polynomial until degree of remainder is less than divisor's degree.

Example 7.1: Divide $x^3 - 6x^2 + 11x - 6$ by $x - 1$ (Long Division)

1. **Divide leading terms:** $\frac{x^3}{x} = x^2$.
   Multiply divisor by x^2: $x^2(x-1) = x^3 - x^2$.
   Subtract:
   $$(x^3 - 6x^2) - (x^3 - x^2) = -5x^2.$$
   Bring down next term $+11x$. New polynomial: $-5x^2 + 11x - 6$.
2. **Divide:** $\frac{-5x^2}{x} = -5x$. Multiply: $-5x(x-1) = -5x^2 + 5x$.
   Subtract:
   $$(-5x^2 + 11x) - (-5x^2 + 5x) = 6x.$$
   Bring down -6. New: $6x - 6$.
3. **Divide:** $\frac{6x}{x} = 6$. Multiply: $6(x-1) = 6x - 6$.
   Subtract:
   $$(6x - 6) - (6x - 6) = 0.$$
4. **Result:** Quotient $x^2 - 5x + 6$, Remainder 0.

Example 7.2: Divide $2x^3 + 3x^2 - x + 5$ by $x^2 + 1$

1. **Divide leading terms:** $\frac{2x^3}{x^2} = 2x$. Multiply: $2x(x^2 + 1) = 2x^3 + 2x$. **Subtract:**

$$(2x^3 + 3x^2 - x) - (2x^3 + 2x) = 3x^2 - 3x.$$

   Bring down $+5$. New: $3x^2 - 3x + 5$.

2. **Divide:** $\frac{3x^2}{x^2} = 3$. Multiply: $3(x^2 + 1) = 3x^2 + 3$. **Subtract:**

$$(3x^2 - 3x + 5) - (3x^2 + 3) = -3x + 2.$$

3. **Result:** Quotient $2x + 3$, Remainder $-3x + 2$.

Example 7.3: Divide $x^4 - x^3 - x + 1$ by $x^2 - 1$

1. **Divide:** $\frac{x^4}{x^2} = x^2$. Multiply: $x^2(x^2 - 1) = x^4 - x^2$.
   Subtract:
   $$(x^4 - x^3) - (x^4 - x^2) = -x^3 + x^2.$$
   Bring down $-x + 1$. New: $-x^3 + x^2 - x + 1$.
2. **Divide:** $\frac{-x^3}{x^2} = -x$. Multiply: $-x(x^2 - 1) = -x^3 + x$.
   Subtract:
   $$(-x^3 + x^2 - x) - (-x^3 + x) = x^2 - 2x.$$
   Bring down $+1$. New: $x^2 - 2x + 1$.
3. **Divide:** $\frac{x^2}{x^2} = 1$. Multiply: $1(x^2 - 1) = x^2 - 1$.
   Subtract:
   $$(x^2 - 2x + 1) - (x^2 - 1) = -2x + 2.$$
4. **Result:** Quotient $x^2 - x + 1$, Remainder $-2x + 2$.

Example 7.7: Divide $x^3 - 6x^2 + 11x - 6$ by $x - 2$ (Synthetic)

Step 1: Identify root $c = 2$.
Step 2: Write coefficients: $1,\ -6,\ 11,\ -6$.

$$\begin{array}{c|cccc} 2 & 1 & -6 & 11 & -6 \\ & & 2 & -8 & 6 \\ \hline & 1 & -4 & 3 & 0 \end{array}$$

Step 3: Quotient coefficients are $1,\ -4,\ 3$, so quotient is $x^2 - 4x + 3$.
Remainder: 0. Hence perfect division.

Example 7.8: Divide $2x^3 + 3x^2 - x + 5$ by $x + 1$

Step 1: Root $c = -1$.
Step 2: Coefficients: $2, \ 3, \ -1, \ 5$.

$$\begin{array}{r|rrrr} -1 & 2 & 3 & -1 & 5 \\ & & -2 & -1 & 2 \\ \hline & 2 & 1 & -2 & 7 \end{array}$$

Step 3: Quotient $2x^2 + x - 2$**, remainder** 7.

Example 7.9: Divide $x^4 - x^3 - x + 1$ by $x - 1$

Step 1: Root $c = 1$**.**
Step 2: Coefficients: $1,\ -1,\ 0,\ -1,\ 1.$

$$\begin{array}{c|ccccc} 1 & 1 & -1 & 0 & -1 & 1 \\ & & 1 & 0 & 0 & -1 \\ \hline & 1 & 0 & 0 & -1 & 0 \end{array}$$

Step 3: Quotient $x^3 - 0x^2 + 0x - 1 = x^3 - 1$**, remainder** 0**.**

Example 7.10: Divide $3x^3 - 2x^2 + x - 5$ by $x - \frac{1}{3}$

Step 1: Root $c = \frac{1}{3}$**.**
Step 2: Coefficients: $3, \ -2, \ 1, \ -5.$

$$\begin{array}{c|cccc} \frac{1}{3} & 3 & -2 & 1 & -5 \\ & & 1 & -\frac{1}{3} & \frac{2}{9} \\ \hline & 3 & -1 & \frac{2}{3} & -\frac{43}{9} \end{array}$$

Step 3: Quotient $3x^2 - x + \frac{2}{3}$**, remainder** $-\frac{43}{9}$**.**

Practice: Polynomial Division (1 of 3)

1. **Divide** $x^3 - 4x^2 + x + 6$ by $x - 2$ (long division).
2. **Divide** $2x^3 + x^2 - 5x + 2$ by $x + 1$ (long division).
3. **Divide** $x^4 - 5x^3 + 6x^2 - 2x + 1$ by $x^2 - 1$ (long division).
4. **Divide** $3x^3 - x^2 + 4x - 8$ by $x - 2$ (synthetic division).
5. **Divide** $x^3 + 2x^2 - x - 2$ by $x + 1$ (synthetic division).

Practice: Polynomial Division (2 of 3)

(6) **Divide** $x^4 - x^3 - x + 1$ by $x - 1$ (synthetic division).

(7) **Divide** $2x^3 - 3x^2 + x - 5$ by $2x - 1$ (synthetic division).

(8) **Divide** $x^3 - 6x^2 + 11x - 6$ by $x - 3$ (synthetic division).

(9) **Divide** $3x^3 + 5x^2 - x + 7$ by $x + 2$ (synthetic division).

(10) **Divide** $2x^4 - x^3 + x^2 - x + 2$ by $x - 1$ (synthetic division).

Practice: Polynomial Division (3 of 3)

(11) **Divide** $x^3 - x^2 - 4x + 4$ by $x - 2$ (long division).

(12) **Divide** $4x^3 - 2x^2 + 3x - 1$ by $2x - 1$ (long division).

(13) **Divide** $x^4 - x^3 + 2x^2 - 3x + 2$ by $x^2 - x + 1$ (long division).

(14) **Divide** $3x^3 - 9x^2 + 6x - 12$ by $x - 2$ (synthetic division).

(15) **Divide** $5x^3 - 5x^2 + 10x - 10$ by $x - 1$ (synthetic division).

Solutions: Polynomial Division

1. Quotient $x^2 - 2x - 3$, Remainder 0.
2. Quotient $2x^2 - x - 4$, Remainder 6.
3. Quotient $x^2 - 4x + 7$, Remainder $6x - 6$.
4. Quotient $3x^2 + 5x + 14$, Remainder 20.
5. Quotient $x^2 + x - 2$, Remainder 0.
6. Quotient $x^3 - x - 1$, Remainder 0.
7. Quotient $2x^2 - 2x$, Remainder -5.
8. Quotient $x^2 - 3x + 2$, Remainder 0.
9. Quotient $3x^2 - x - 3$, Remainder 1.
10. Quotient $2x^3 + x^2 + 2x + 1$, Remainder 3.
11. Quotient $x^2 + x - 2$, Remainder 0.
12. Quotient $2x^2 - \frac{x}{2} + \frac{3}{4}$, Remainder $\frac{5}{4}$.
13. Quotient $x^2 - x + 3$, Remainder -1.
14. Quotient $3x^2 - 3x$, Remainder -12.
15. Quotient $5x^2$, Remainder 0.

Remainder and Factor Theorems

- **Remainder Theorem:** When $f(x)$ is divided by $(x - c)$, remainder is $f(c)$.
- **Factor Theorem:** $(x - c)$ is factor of $f(x)$ if and only if $f(c) = 0$.

Example 8.1: Remainder of $x^3 - 2x^2 + x - 5$ by $(x - 2)$

$$f(x) = x^3 - 2x^2 + x - 5, \quad c = 2.$$

Compute $f(2)$:

$$f(2) = 2^3 - 2 \cdot 2^2 + 2 - 5 = 8 - 8 + 2 - 5 = -3.$$

Result: Remainder is -3.

Example 8.2: Show $(x+3)$ is factor of $x^3+6x^2+11x+6$

$$g(x) = x^3 + 6x^2 + 11x + 6, \quad c = -3 \text{ for } x + 3.$$

Compute $g(-3)$:

$$g(-3) = (-3)^3 + 6(-3)^2 + 11(-3) + 6 = -27 + 54 - 33 + 6 = 0.$$

Since $g(-3) = 0$, $(x+3)$ is a factor.

Example 8.3: Find a if $2x^3 - 3x^2 + ax - 5$ leaves remainder 7 by $(x - 1)$

$$h(x) = 2x^3 - 3x^2 + ax - 5, \quad c = 1.$$

We require $h(1) = 7$:

$$h(1) = 2(1)^3 - 3(1)^2 + a(1) - 5 = 2 - 3 + a - 5 = -6 + a.$$

Set $-6 + a = 7 \implies a = 13.$

Example 8.4: Factorize $x^3 - 4x^2 + x + 6$ given $(x - 2)$ is a factor

$$f(x) = x^3 - 4x^2 + x + 6, \quad c = 2.$$

Check $f(2)$:

$$f(2) = 8 - 16 + 2 + 6 = 0,$$

so $(x - 2)$ is indeed a factor. Perform synthetic division or long division:

$$\begin{array}{c|cccc} 2 & 1 & -4 & 1 & 6 \\ & & 2 & -4 & -6 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

Quotient: $x^2 - 2x - 3$. Factor quotient:

$$x^2 - 2x - 3 = (x - 3)(x + 1).$$

Therefore,

$$x^3 - 4x^2 + x + 6 = (x - 2)(x - 3)(x + 1).$$

Example 8.5: If $x^3 + px^2 + qx + r$ is divisible by $(x-1)$ and $(x+2)$

$$p(x) = x^3 + px^2 + qx + r.$$

Step 1: Use remainder theorem:

- Divisible by $(x-1) \implies p(1) = 0$. So:

$$1 + p + q + r = 0 \quad (1).$$

- Divisible by $(x+2) \implies p(-2) = 0$. So:

$$(-2)^3 + p(-2)^2 + q(-2) + r = -8 + 4p - 2q + r = 0 \quad (2).$$

Step 2: Subtract (2) - (1):

$$[-8+4p-2q+r]-[1+p+q+r] = 0-0 \implies -9+3p-3q = 0 \implies 3p-3q = 9$$

Thus, $p - q = 3$ is the relationship between p and q.

Example 8.6: Find k such that $(x-2)$ is a factor of $kx^3 - x^2 + 4x - 8$

$$f(x) = kx^3 - x^2 + 4x - 8, \quad c = 2.$$

Set $f(2) = 0$:

$$f(2) = 8k - 4 + 8 - 8 = 8k - 4.$$

Solve $8k - 4 = 0 \implies k = \frac{1}{2}$.

Practice: Remainder/Factor (1 of 2)

1. **Find** the remainder when $f(x) = x^3 + x^2 - x - 1$ is divided by $(x - 1)$.
2. **Determine** if $(x + 2)$ is a factor of $g(x) = x^3 + x^2 - 4x - 4$.
3. **Find** a if $h(x) = 2x^3 + ax^2 - 5x + 3$ leaves remainder 4 when divided by $(x - 1)$.
4. **Factorize** $f(x) = x^3 - 6x^2 + 11x - 6$ using the Factor Theorem.
5. **Given** $p(x) = x^3 + px^2 + qx + r$ is divisible by $(x - 2)$ and $(x + 1)$, find relationships among p, q, r.

(6) **Find** k such that $(x-3)$ is a factor of $3x^3 + kx^2 - 9x + 27$.

(7) **Find** the remainder when $f(x) = 5x^4 - 2x^3 + x - 1$ is divided by $(x+1)$.

(8) **Determine** if $(x-4)$ is a factor of $g(x) = x^3 - 5x^2 + 8x - 20$.

(9) **Find** a, b such that $(x-1)$ and $(x+2)$ are factors of $h(x) = x^3 + ax^2 + bx - 4$.

(10) **Factorize** $f(x) = 2x^3 - x^2 - 2x + 1$ given that $(x - \frac{1}{2})$ is a factor.

Solutions: Remainder/Factor

1. $f(1) = 1 + 1 - 1 - 1 = 0$. Remainder 0.
2. $g(-2) = -8 + 4 + 8 - 4 = 0$. Yes, $(x + 2)$ is a factor.
3. $h(1) = 2 + a - 5 + 3 = a = 4$. So $a = 4$.
4. Roots: $1, 2, 3$. So $f(x) = (x - 1)(x - 2)(x - 3)$.
5. Conditions: $1 + p + q + r = 0$, $-1 + p - q + r = 0$. From system: $p + q + r = -1$, $p - q + r = 1$. Subtract: $2q = -2 \implies q = -1$, then $p + r - 1 = -1 \implies p + r = 0$.
6. $f(3) = 81 + 9k - 27 + 27 = 81 + 9k$. Set $= 0 \implies 9k = -81 \implies k = -9$.
7. $f(-1) = 5 - 2 - 1 - 1 = 1$. Remainder 1.
8. $g(4) = 64 - 80 + 32 - 20 = -4 \neq 0$. Not a factor.
9. $h(1) = 1 + a + b - 4 = 0$, $h(-2) = -8 + 4a - 2b - 4 = 0$. Solve: $a + b = 3$, $4a - 2b = 12 \implies 2a - b = 6$. So $a = 3, b = 0$.
10. $f(\frac{1}{2}) = 2(\frac{1}{8}) - \frac{1}{4} - \frac{1}{2} + 1 = \frac{1}{4} - \frac{1}{4} - \frac{1}{2} + 1 = \frac{1}{2}$. Since remainder nonzero, misprint. Should check divisor $(2x - 1)$: root $\frac{1}{2}$ gives remainder 0. Factor: $2x - 1$. Then quotient is $x^2 - \frac{x}{2} - 1$.