- MRS (Marginal Rate of Substitution): is the slope of the indifference curve.

$$MRS = \left(\frac{dy}{dx}\right)_{U\ constant} = -\frac{\left(\frac{\partial U}{\partial x}\right)}{\left(\frac{\partial U}{\partial y}\right)} = -\left(\frac{MgU_x}{MgU_y}\right)$$

- ERS (Economic Rate of Substitution): is the slope of the budget constraint.

$$ERS = -\frac{p_x}{p_y}$$

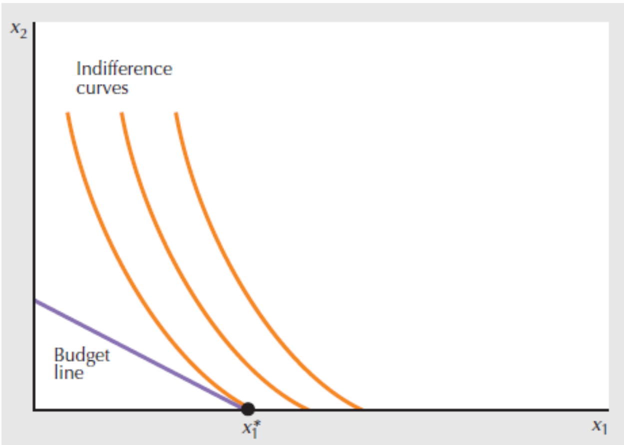
In the optimal choice, we have:

$$MRS = ERS$$

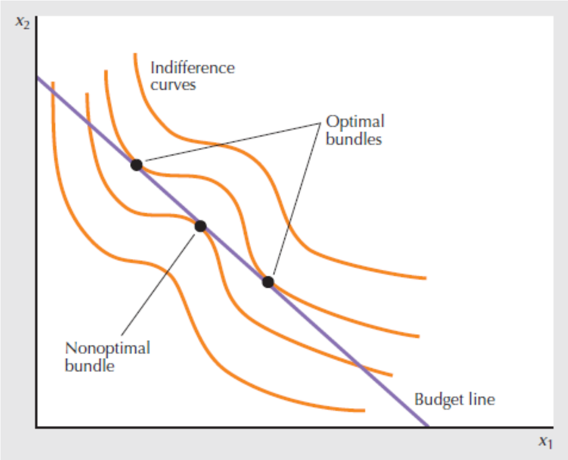
$$-\frac{MgU_x}{MgU_y} = -\frac{p_x}{p_y}$$

$$\boldsymbol{\frac{MgU_x}{MgU_y} = \frac{p_x}{p_y}}$$

This condition above is a necessary condition for the optimal choice in the cases in which the solution is in the interior of the graph. When the solution is in the boundary, then this condition doesn't apply.



This a necessary condition, but not a sufficient one. In the graph below, there are three points in which the condition is satisfied, but only two are optimal choices.



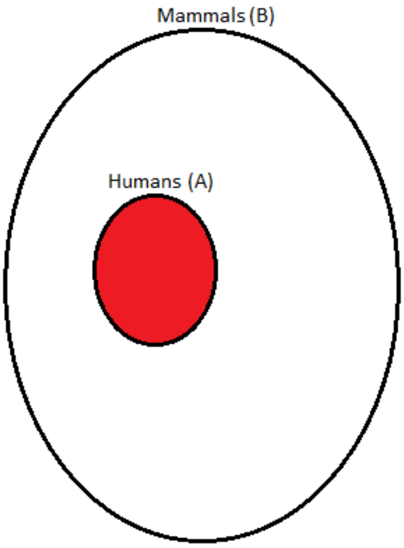
There is just one case in which this condition is sufficient: when the indifference curves are well-behaved (that is, they are monotonic and convex).

Logic Section

- $A \to B$:
  - A is a **sufficient condition** for B.
  - B is a **necessary condition** for A.

Example: A is human, B is mammal.

- If an animal is a human, then he is a mammal. Then A is a sufficient condition for B.
- For an animal to be a human, he needs to be a mammal. Then B is a necessary condition for A.



$$U(x_1, x_2) = x_1^c x_2^d$$

- $\left(\frac{c}{c+d}\right)$: the share of income spent with good x_1.
- $\left(\frac{d}{c+d}\right)$: the share of income spent with good x_2.

Particularly, if $c + d = 1$, then c is the share of income spent with good x_1 and d is the share of income spent with good x_2.

1º)

$$MgU_{x_1} = \frac{\partial U}{\partial x_1} = c x_1^{c-1} x_2^{d}$$

$$MgU_{x_2} = \frac{\partial U}{\partial x_2} = d x_1^{c} x_2^{d-1}$$

2º)

$$\frac{MgU_{x_1}}{MgU_{x_2}} = \frac{p_{x_1}}{p_{x_2}}$$

$$\frac{\left(c x_1^{c-1} x_2^{d}\right)}{\left(d x_1^{c} x_2^{d-1}\right)} = \frac{p_{x_1}}{p_{x_2}}$$

Objective: Isolate x_2.

$$\left(\frac{c}{d}\right)\left(\frac{x_2^{d-d+1}}{x_1^{c-c+1}}\right) = \frac{p_{x_1}}{p_{x_2}}$$

$$\left(\frac{c}{d}\right)\left(\frac{x_2}{x_1}\right) = \frac{p_{x_1}}{p_{x_2}}$$

$$\boldsymbol{x_2 = \left(\frac{d}{c}\right)\left(\frac{p_{x_1}}{p_{x_2}}\right) x_1} \quad \textbf{(Equation 1)}$$

3º) Substitute equation (1) in the budget constraint:

$$p_{x_1} x_1 + p_{x_2} x_2 = m$$

$$p_{x_1} x_1 + p_{x_2} \overbrace{\left[\left(\frac{d}{c}\right)\left(\frac{p_{x_1}}{p_{x_2}}\right) x_1\right]}^{x_2} = m$$

Objective: Isolate x_1.

$$p_{x_1} x_1 + \left(\frac{d}{c}\right) p_{x_1} x_1 = m$$

$$p_{x_1}\left[x_1 + \left(\frac{d}{c}\right) x_1\right] = m$$

$$x_1 + \left(\frac{d}{c}\right) x_1 = \frac{m}{p_{x_1}}$$

$$x_1 \left(1 + \frac{d}{c}\right) = \frac{m}{p_{x_1}}$$

$$x_1 \left(\frac{c}{c} + \frac{d}{c}\right) = \frac{m}{p_{x_1}}$$

$$x_1 \left(\frac{c + d}{c}\right) = \frac{m}{p_{x_1}}$$

$$\boldsymbol{x_1^* = \left(\frac{c}{c + d}\right)\left(\frac{m}{p_{x_1}}\right)}$$

4º) Substitute x_1^* in equation (1):

$$x_2^* = \left(\frac{d}{c}\right)\left(\frac{p_{x_1}}{p_{x_2}}\right) x_1^*$$

$$x_2^* = \left(\frac{d}{c}\right)\left(\frac{p_{x_1}}{p_{x_2}}\right)\overbrace{\left[\left(\frac{c}{c + d}\right)\left(\frac{m}{p_{x_1}}\right)\right]}^{x_1^*}$$

$$\boldsymbol{x_2 = \left(\frac{d}{c + d}\right)\left(\frac{m}{p_2}\right)}$$

Exercises

Question 1

$$U(x_1, x_2) = x_1 + x_2$$

$$x_2 = \begin{cases} \frac{m}{p_2}, & if\ p_1 > p_2 \\ \left[0, \frac{m}{p_2}\right], & if\ p_1 = p_2 \\ 0, & if\ p_1 < p_2 \end{cases}$$

Question 2

$$MRS = -b = -\frac{b}{1}$$

$$MRS = -\frac{MgU_1}{MgU_2}$$

Equalizing both equations above:

$$-\frac{b}{1} = -\frac{MgU_1}{MgU_2}$$

$$MgU_1 = b; \;\; MgU_2 = 1$$

$$\frac{\partial U}{\partial x_1} = b; \frac{\partial U}{\partial x_2} = 1$$

Then the utility function is:

$$\boldsymbol{U(x_1, x_2) = bx_1 + x_2}$$

Demand function:

$$\frac{\overbrace{MgU_1}^{b}}{p_1} > \frac{\overbrace{MgU_2}^{1}}{p_2} \rightarrow x_1 = \frac{m}{p_1}$$

$$\frac{b}{p_1} > \frac{1}{p_2} \rightarrow x_1 = \frac{m}{p_1}$$

$$\frac{b}{1} > \frac{p_1}{p_2} \rightarrow x_1 = \frac{m}{p_1}$$

Synthesis:

$$\boldsymbol{x_1(p_1, p_2, m) = \begin{cases} 0, \;\; if \; \frac{p_1}{p_2} > b \\ \left[0, \frac{m}{p_1}\right], \;\; if \; \frac{p_1}{p_2} = b \\ \frac{m}{p_1}, \;\; if \; \frac{p_1}{p_2} < b \end{cases}}$$

$$\boldsymbol{x_2(p_1, p_2, m) = \begin{cases} \frac{m}{p_2}, \;\; if \; \frac{p_1}{p_2} > b \\ \left[0, \frac{m}{p_2}\right], \;\; if \; \frac{p_1}{p_2} = b \\ 0, \;\; if \; \frac{p_1}{p_2} < b \end{cases}}$$

Question 3

$$U(s, c) \; = min\{2s, c\}$$

1º)

$$2s = c$$

$$p_s s + p_c c = m$$

$$p_s s + p_c(2s) = m$$

$$s(p_s + 2p_c) = m$$

$$\boldsymbol{s(p_s, p_c, m) = \frac{m}{p_s + 2p_c}}$$

2º)

$$2s = c \rightarrow s = \frac{c}{2}$$

$$p_s \overbrace{\left(\frac{c}{2}\right)}^{s} + p_c c = m$$

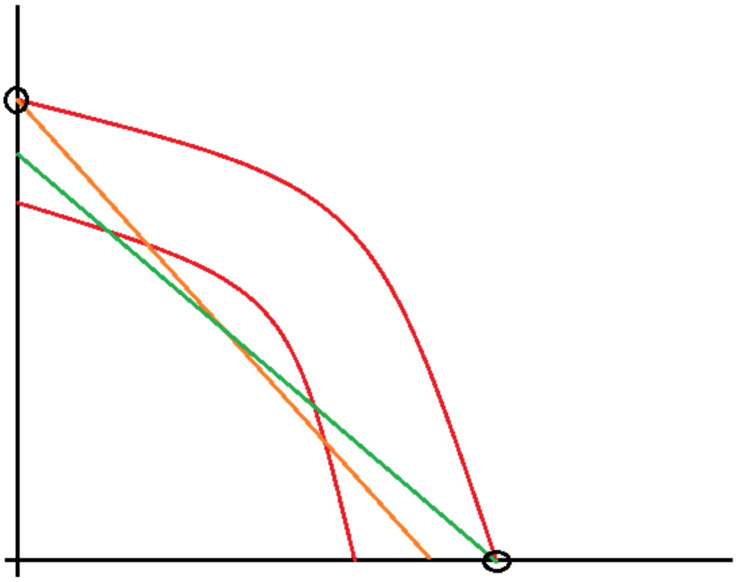
$$c\left(\frac{p_s}{2} + p_c\right) = m$$

$$c\left(\frac{p_s + 2p_c}{2}\right) = m$$

$$\boldsymbol{c(p_s, p_c, m) = \frac{2m}{p_s + 2p_c}}$$

Question 4

The individual's choice would depend on the slope of the budget constraint, that is, on the ratio $\frac{p_1}{p_2}$.



Question 5

$$U(x_1, x_2) = x_1^1 x_2^4$$

1º) Finding the demand for each good:

$$x_1 = \left(\frac{1}{1+4}\right)\left(\frac{m}{p_1}\right) \rightarrow \boldsymbol{x_1 = \left(\frac{1}{5}\right)\left(\frac{m}{p_1}\right)}$$

$$x_2 = \left(\frac{4}{1+4}\right)\left(\frac{m}{p_2}\right) \rightarrow \boldsymbol{x_2 = \left(\frac{4}{5}\right)\left(\frac{m}{p_2}\right)}$$

2º) Finding the share (s) of income spent on each good:

The share spent on good i is:

$$s_i = \frac{p_i x_i}{m}$$

Substituting the demand functions on the equation above:

$$s_1 = \frac{p_1}{m}\overbrace{\left[\left(\frac{1}{5}\right)\left(\frac{m}{p_1}\right)\right]}^{x_1}$$

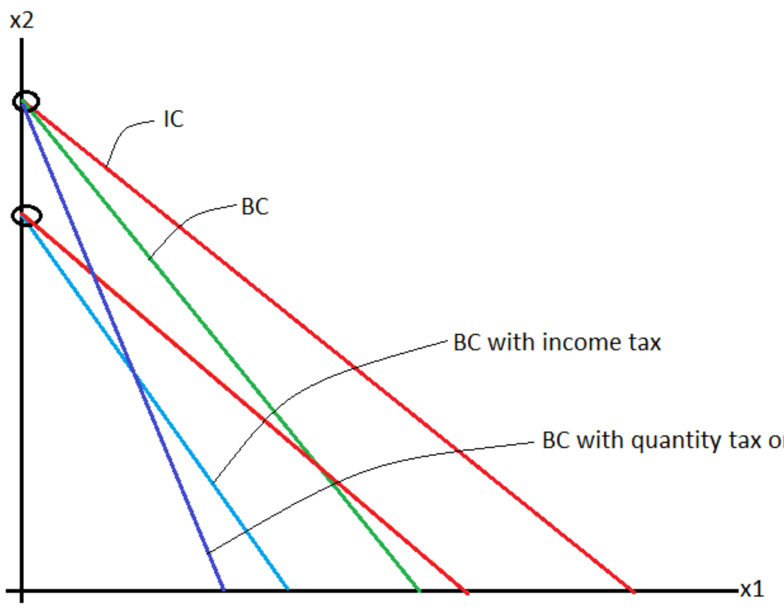
$$\boldsymbol{s_1 = \frac{1}{5}}$$

$$s_2 = \frac{p_2}{m}\overbrace{\left[\left(\frac{4}{5}\right)\left(\frac{m}{p_2}\right)\right]}^{x_2}$$

$$\boldsymbol{s_2 = \frac{4}{5}}$$

Question 6

Perfect Substitutes: The condition that a quantity tax and an income tax generate the same level of utility is not satisfied, as can be seen in the graph below. The red lines are the indifference curve, the green line is the original budget constraint, the light blue line is the budget constraint after an income tax and the dark blue is the budget constraint after a quantity tax.



Perfect Complements: the line colors are the same as before.