Calculus AI-SL Pt.1

- Intro to derivatives
- First-order derivative

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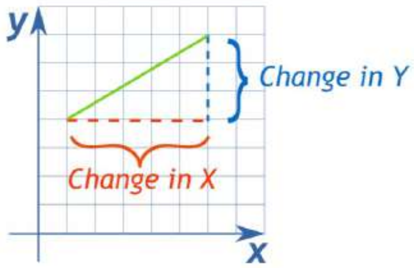
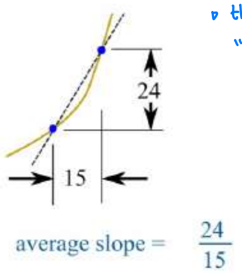
Intro to Derivatives

It is all about slope!

Slope = $\dfrac{\text{Change in Y}}{\text{Change in X}}$



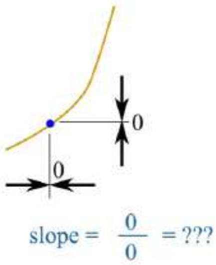
We can find an **average** slope between two points.



this is called "secant line"

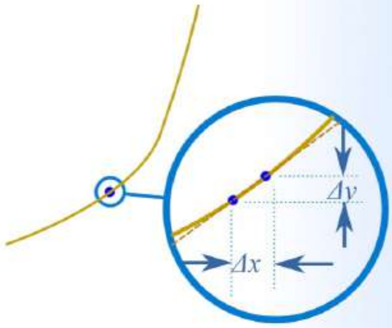
But how do we find the slope **at a point**?

There is nothing to measure!



But with derivatives we use a small difference ...

... then have it **shrink towards zero**.



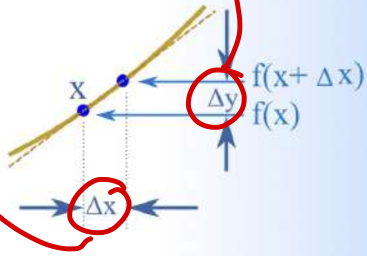
{The key point of derivative is that we want to find a (tangent) gradient at a point from f(x) }

Let us Find a Derivative!

To find the derivative of a function $y = f(x)$ we use the slope formula:

$$\text{Slope} = \frac{\text{Change in Y}}{\text{Change in X}} = \frac{\Delta y}{\Delta x}$$

And (from the diagram) we see that:



x changes from x to $x+\Delta x$

y changes from $f(x)$ to $f(x+\Delta x)$

Now follow these steps:

- Fill in this slope formula: $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$
- Simplify it as best we can
- Then make **Δx** shrink towards zero.

Notation

"Shrink towards zero" is actually written as a limit like this:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

"The derivative of **f** equals

the limit as Δx goes to zero of $f(x+\Delta x)$ - $f(x)$ over Δx"

Example: the function $f(x) = x^2$

We know $\mathbf{f(x) = x^2}$, and we can calculate $\mathbf{f(x+\Delta x)}$:

Start with: $\mathbf{f(x+\Delta x) = (x+\Delta x)^2}$

Expand $(x + \Delta x)^2$: $\mathbf{f(x+\Delta x) = x^2 + 2x\,\Delta x + (\Delta x)^2}$

The slope formula is: $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Put in $\mathbf{f(x+\Delta x)}$ and $\mathbf{f(x)}$: $\dfrac{x^2 + 2x\,\Delta x + (\Delta x)^2 - x^2}{\Delta x}$

Simplify (x^2 and $-x^2$ cancel): $\dfrac{2x\,\Delta x + (\Delta x)^2}{\Delta x}$

Simplify more (divide through by Δx): $= 2x + \Delta x$

Then, **as Δx heads towards 0** we get: $= 2x$

Result: the derivative of $\mathbf{x^2}$ is $\mathbf{2x}$

In other words, the slope at x is **2x**

just FYI, no need to remember

We write **dx** instead of **"Δx heads towards 0"**.

And "the derivative of" is commonly written $\frac{d}{dx}$ like this:

$$\frac{d}{dx}x^2 = 2x$$

"The derivative of $\mathbf{x^2}$ *equals* **2x**"

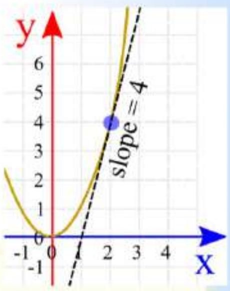
or simply *"d dx of* $\mathbf{x^2}$ *equals* **2x**"

So what does $\frac{d}{dx}x^2 = 2x$ mean?

It means that, for the function x^2, the slope or $\boxed{\text{"rate of change"}}$ at any point is **2x**.

instantaneous

So when **x=2** the slope is **2x = 4**, as shown here:



Or when **x=5** the slope is **2x = 10**, and so on.

Note: $f'(x)$ can also be used for "the derivative of":

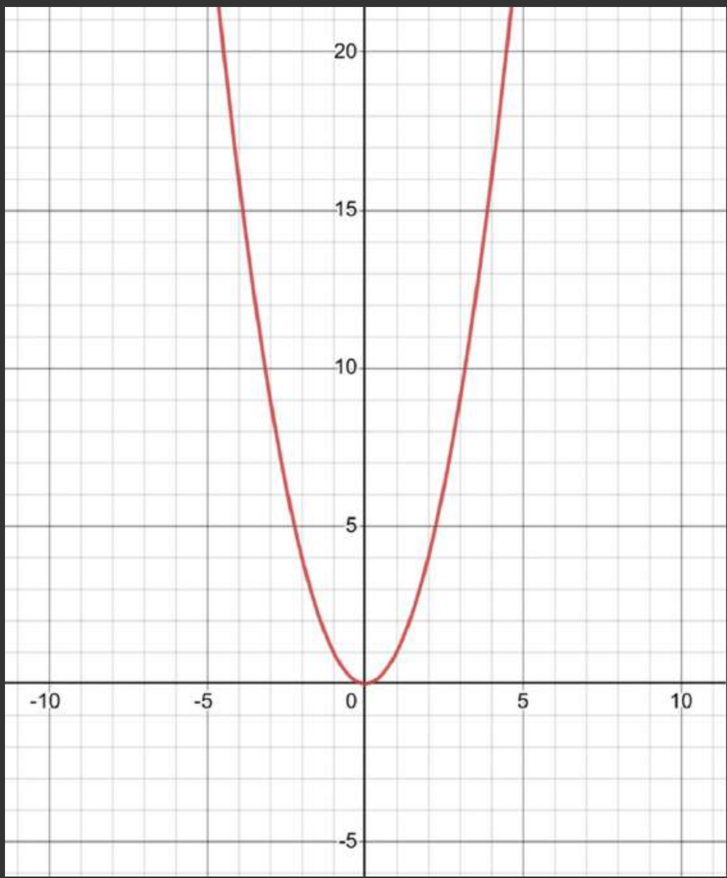
$$f'(x) = 2x$$

"The derivative of f(x) equals 2x"

or simply *"f-dash of x equals 2x"*

First - order Derivative

Let's say we have $f(x) = x^2$



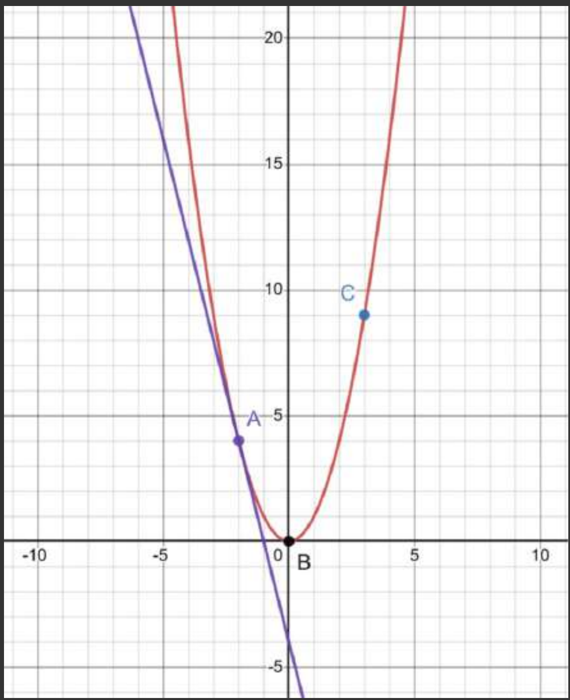
And we want to know the

GRADIENT at point

A: $(-2, 4)$

$f(-2) = (-2)^2$

$= 4$



As $f'(x) = 2x$,

the slope at A is :

$$m = f'(-2)$$
$$= 2(-2)$$
$$\therefore m = -4$$

With a slope (m = −4) & point, A : (−2, 4)

we can make a line.

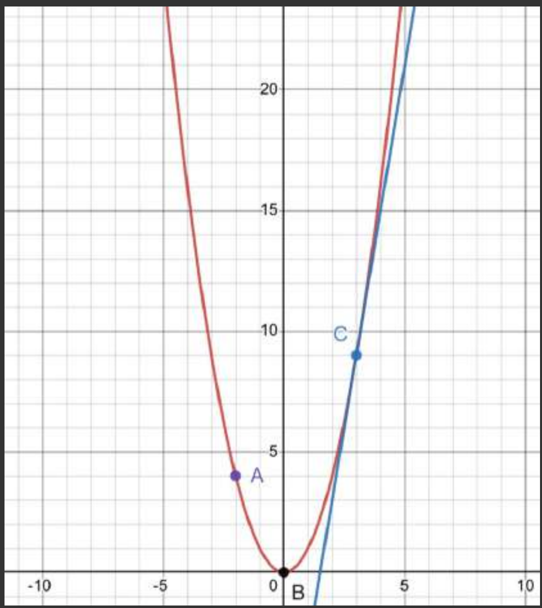
$$y - y_1 = \underset{2\cdot -2}{m}(x - x_1)$$

$$y - 4 = -4(x - (-2))$$

$$y - 4 = -4(x + 2)$$

$$\therefore y = -4x - 4$$

* this is the TANGENT LINE of $f(x)$ at $x = -2$



Let's say we want:

(i) instantaneous rate of change, and

(ii) tangent line eq.

of $f(x) = x^2$ at $x = 3$, i.e point C.

(i) Instantaneous rate of change (or slope / gradient) requires first-order derivative.

$$f(x) = x^2$$
$$f'(x) = 2x$$

At C, slope is:

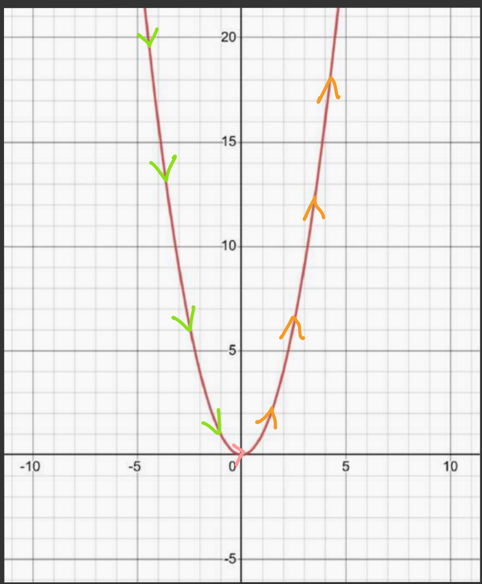
$$m = f'(3) = 2\ 3$$
$$\therefore m = 6$$

(ii) To make tangent line eq., we need a slope & a point.

* $\underline{m_t = 6}$ [from (i)]
* At $x = 3$, $y = f(3) = 3^2 = 9$. $\underline{C : (3, 9)}$

$$y - y_1 = m\,(x - x_1)$$
$$\therefore y - 9 = 6\,(x - 3)$$ * you may stop here
$$y - 9 = 6x - 18$$
$$\therefore y = 6x - 9$$ * or here

Notice that when we want to make a tangent line,

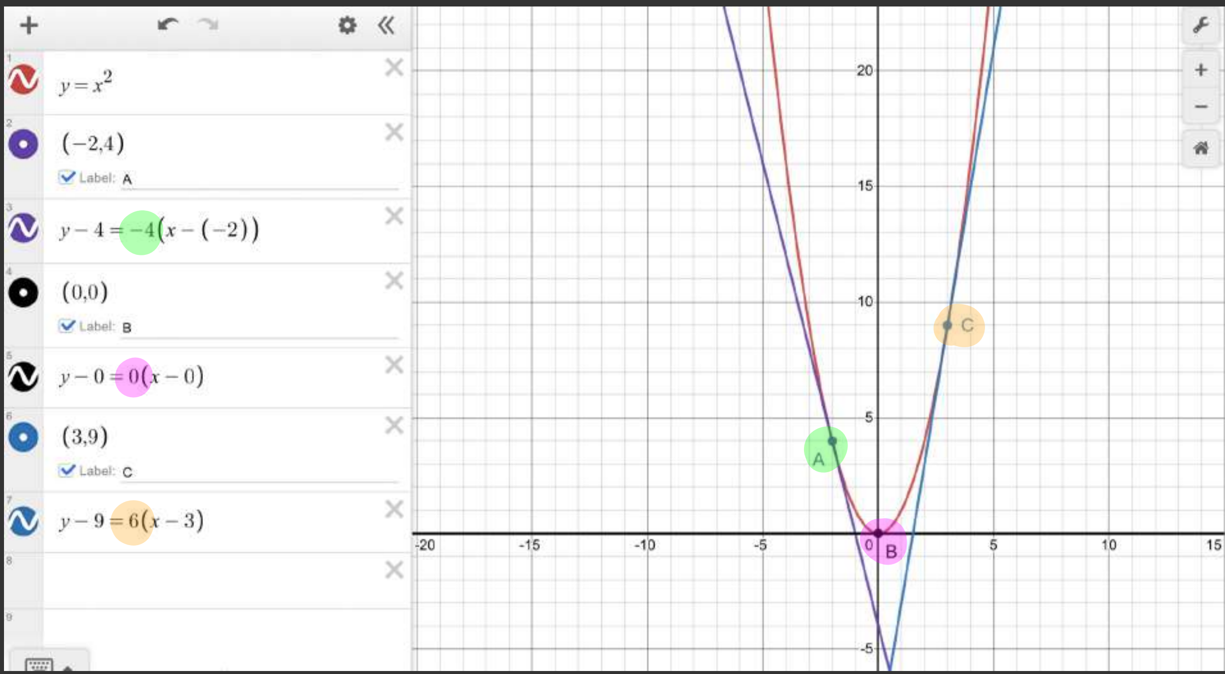


the SLOPE of $f(x)$ at :

* $x < 0$ must be negative
* $x = 0$ must be 0.
* $x > 0$ must be positive

This aligns with nature of $f(x)$:

* $f(x)$ is decreasing at $x < 0$
* $f(x)$ is stationary at $x = 0$
* $f(x)$ is increasing at $x > 0$



so that the connection between $f'(x)$ and $f(x)$ is

when	function is
$f'(x) < 0$	decreasing
$f'(x) = 0$	stationary (local min. / max.)
$f'(x) > 0$	increasing

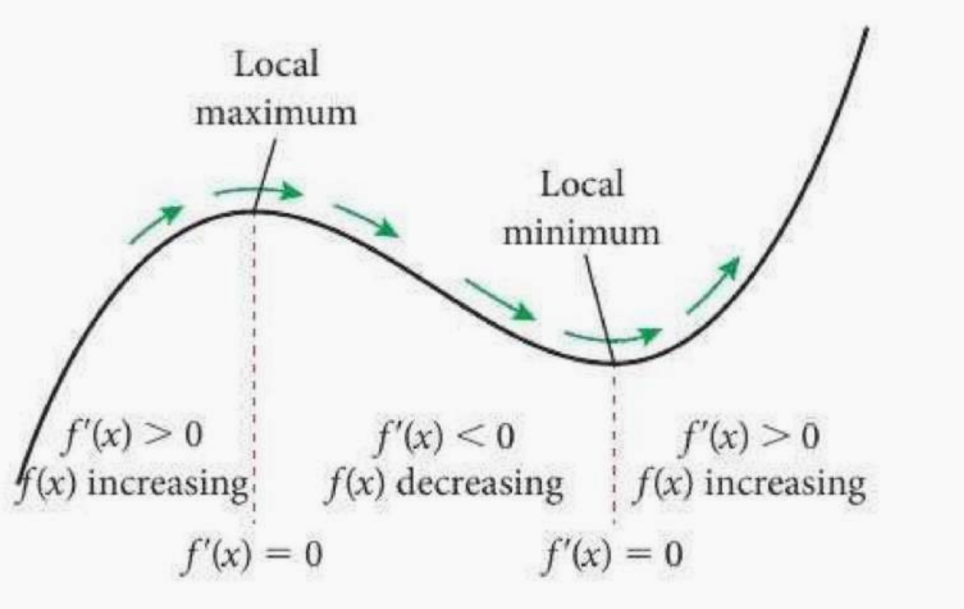
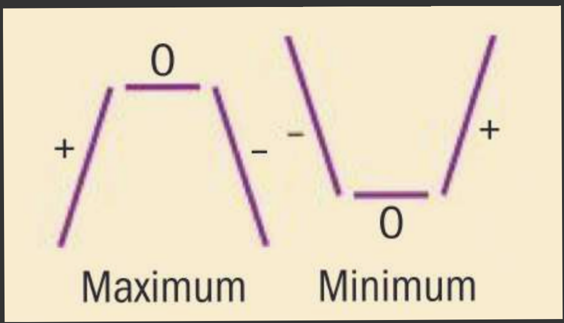


Figure 10.2 Local extrema



Practice FO Derivative

Find the (i) derivative, (ii) gradient, (iii) tangent eq. of the functions below at $x = 2$.

1 $y = 3x^4$	**2** $y = 5x^3$	**3** $y = 2x^2 + 3x - 5$
4 $y = x^3 + 2x + 1$	**5** $y = \frac{2}{x} + x, x \neq 0$	**6** $y = \frac{3}{x^2}, x \neq 0$

And state:

(iv) the function is increasing/ decreasing at $x = 2$

(v) the range of values when function is increasing & decreasing.

1. (i) $y = 3x^4$

$y' = 4 \cdot 3x^3$ power rule "DOMO"

$\therefore y' = 12x^3$

(ii) $m = f'(2)$

$= 12 \cdot 2^3$

$\therefore m = 96$

(iii) * $m_t = 96$ → from (ii)

* At $x = 2$, $y = f(2) = 3\ 2^4 = 48$

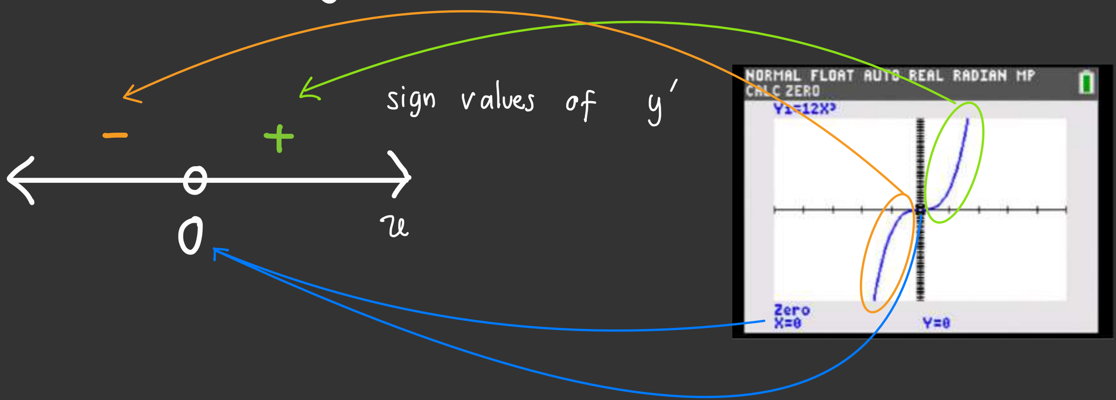
$P : (2, 48)$

$y - y_p = m_t(x - x_p)$

$\therefore y - 48 = 96(x - 2)$

(iv) As $f'(2) = 96 > 0$, function is increasing at $x = 2$.

(v) Visualising $y' = 12x^3$ with sign values :



$\therefore f$ is increasing at $x > 0$, and decreasing at $x < 0$

3. (i) $y = 2x^2 + 3x - 5$

$y' = 2 \cdot 2x + 3$

$\therefore y' = 4x + 3$

(ii) $m = f'(2)$

$= 4(2) + 3$

$\therefore m = 11$

(iii) * $m_t = 11$ → from (ii)

* At $x = 2$, $y = f(2) = 2 \cdot 2^2 + 3 \cdot 2 - 5 = 9$

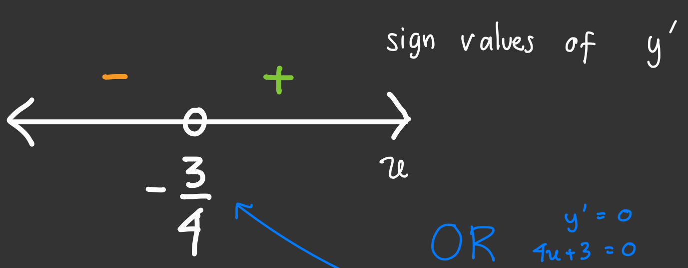
$P : (2, 9)$

$y - y_p = m_t(x - x_p)$

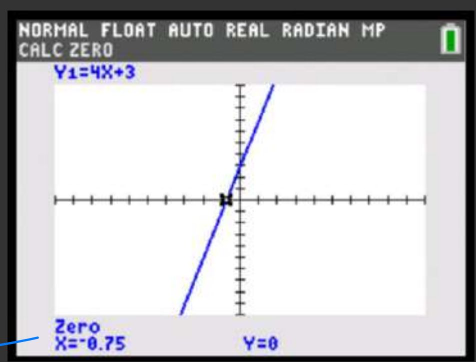
$\therefore y - 9 = 11(x - 2)$

(iv) As $f'(2) = 11 > 0$, function is increasing at $x = 2$.

(v) Visualising $y' = 4x + 3$ with sign values :



OR
$y' = 0$
$4x + 3 = 0$
$4x = -3$
$x = -3/4$



$\therefore f$ is increasing at $x > -\frac{3}{4}$, and decreasing at $x < -\frac{3}{4}$

5. (i) $y = 2x^{-1} + x$

$y' = -1 \cdot 2x^{-2} + 1$

$\therefore y' = -\dfrac{2}{x^2} + 1 \quad [\underbrace{x \neq 0}_{\text{asymptote}}]$

(ii) $m = f'(2)$

$= -\dfrac{2}{2^2} + 1$

$\therefore m = \dfrac{1}{2}$

(iii) * $m_t = \dfrac{1}{2}$ → from (ii)

* At $x = 2$, $y = f(2) = \dfrac{2}{2} + 2 = 3$

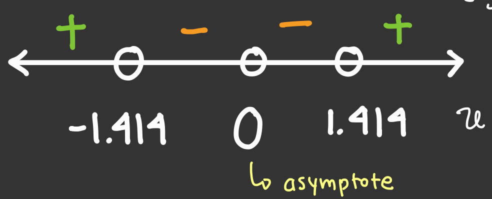
$P : (2, 3)$

$y - y_p = m_t(x - x_p)$

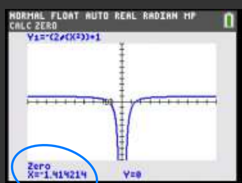
$\therefore y - 3 = \dfrac{1}{2}(x - 2)$

(iv) As $f'(2) = \dfrac{1}{2} > 0$, function is increasing at $x = 2$.

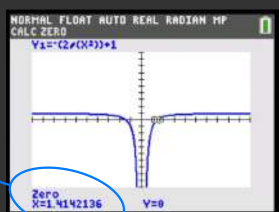
(v) Visualising $y' = -\dfrac{2}{x^2} + 1$ with sign values :



sign values of y'



OR



$y' = 0$

$-\dfrac{2}{x^2} + 1 = 0$

$-\dfrac{2}{x^2} = -1$

$2 = x^2$

$\therefore x = \pm\sqrt{2}$

$\therefore f$ is increasing at $x < -1.414$ and $x > 1.414$,

decreasing at $-1.414 < x < 1.414$ with $x \neq 0$.