Utility Function

Suppose that you prefer $x = \{x_1, x_2\}$ over $y = \{y_1, y_2\}$. That is, $x \succ y$. Then a utility function is a function such that:

$$U(x) > U(y)$$

Monotonic Transformation

Monotonic transformations are transformations in the utility function that don't alter the order of the individual's bundle preferences.

Suppose the following utility function:

$$U(x_1, x_2) = x_1 x_2$$

Now the following transformation is done:

$$f[U(x_1, x_2)] = \ln[U(x_1, x_2)] = \ln \overbrace{(x_1 x_2)}^{U(x_1, x_2)}$$
$$= \ln x_1 + \ln x_2$$

This is a monotonic transformation, because for every two pairs of bundles, x' and x'', such that $U(x') > U(x'')$ we have:

$$f[U(x')] > f[U(x'')]$$

Conclusion: $U(x_1, x_2) = x_1 x_2$ and $\overbrace{V(x_1, x_2)}^{f[U(x_1, x_2)]} = \ln x_1 + \ln x_2$ represent the same preferences.

Common monotonic transformations:

$$\pm\ constant;\ /\ or\ \times\ by\ a\ constant;\ ln;\ e$$

- $e = 2{,}71 \ldots$

Deriving Indifference Curves from Utility

To derive indifference curves from utility functions, we do the following procedure:

1. Let $u(x_1, x_2)$ be called k, that is, $u(x_1, x_2) \equiv k$.
2. Isolate x_2, making x_2 as function of x_1 and k.
3. For every value of k, there is a different indifference curve.

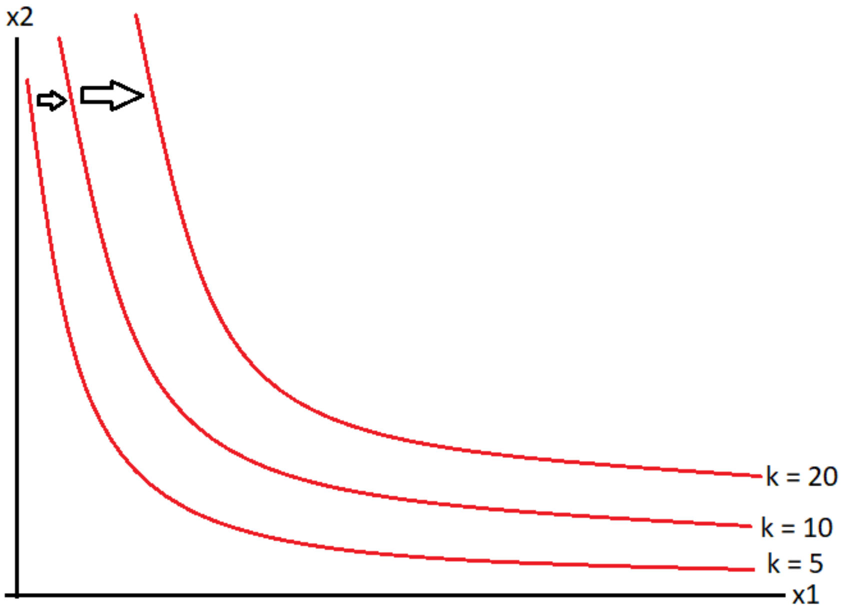
Examples:

$$\overbrace{u(x_1, x_2)}^{k} = x_1 x_2$$

$$\boldsymbol{x_2 = \frac{k}{x_1}}$$

$$\overbrace{u(x_1, x_2)}^{k} = x_1^{\alpha} x_2^{\beta}$$

$$\boldsymbol{x_2 = \left(\frac{k}{x_1^{\alpha}}\right)^{\frac{1}{\beta}}}$$



Perfect Complements

$$\overbrace{u(x_1, x_2)}^{k} = \min\{ax_1, bx_2\}$$

1º) $ax_1 > bx_2$:

$$k = bx_2$$

$$x_2 = \frac{k}{b}$$

2º) $ax_1 < bx_2$:

$$k = ax_1$$

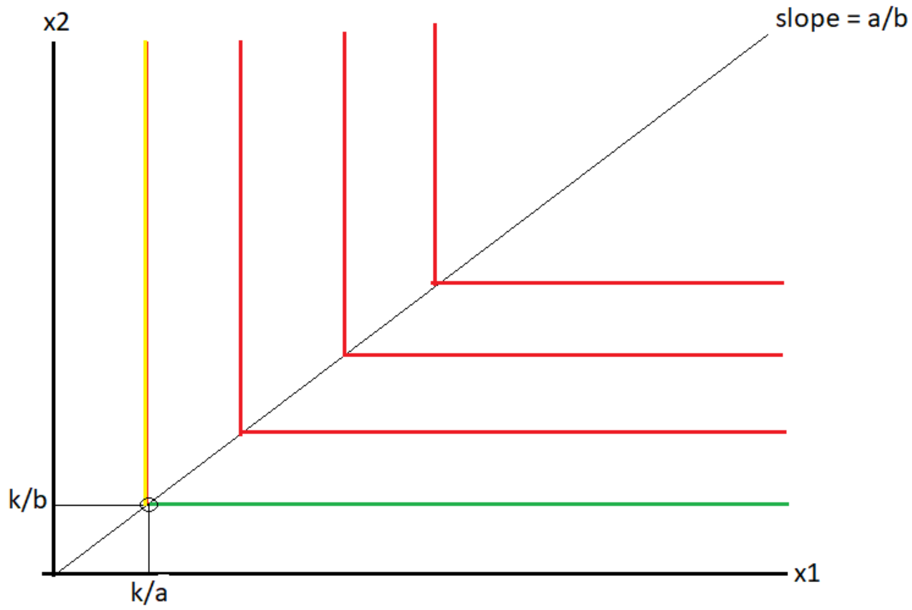
$$x_1 = \frac{k}{a}$$

3º) $ax_1 = bx_2$:

$$k = ax_1 = bx_2$$

$$ax_1 = bx_2$$

$$\boldsymbol{x_2 = \left(\frac{a}{b}\right) x_1}$$



Cobb-Douglas Utility Functions

$$u(x_1, x_2) = x_1^c x_2^d$$

$$F(L, K) = L^c K^d$$

- $c + d = 1$: **Constant returns to scale:** if the firm doubles the inputs (L and K), the production increases by a factor of 2.
- $c + d < 1$: **Decreasing returns to scale:** if the firm doubles the inputs (L and K), the production increases by a factor less than 2.
- $c + d > 1$: **Increasing returns to scale**: if the firm doubles the inputs (L and K), the production increases by a factor more than 2.

Deriving the formula of MRS from the total derivative

Suppose we apply a **total derivative** to $u(x_1, x_2)$, then we have:

$$dU = \overbrace{\frac{\partial U}{\partial x_1}}^{MU_1} dx_1 + \overbrace{\frac{\partial U}{\partial x_2}}^{MU_2} dx_2$$

The MRS is calculated on an indifference curve, so we have to have $dU = 0$. Then:

$$0 = MU_1 dx_1 + MU_2 dx_2$$

$$MU_2 dx_2 = -MU_1 dx_1$$

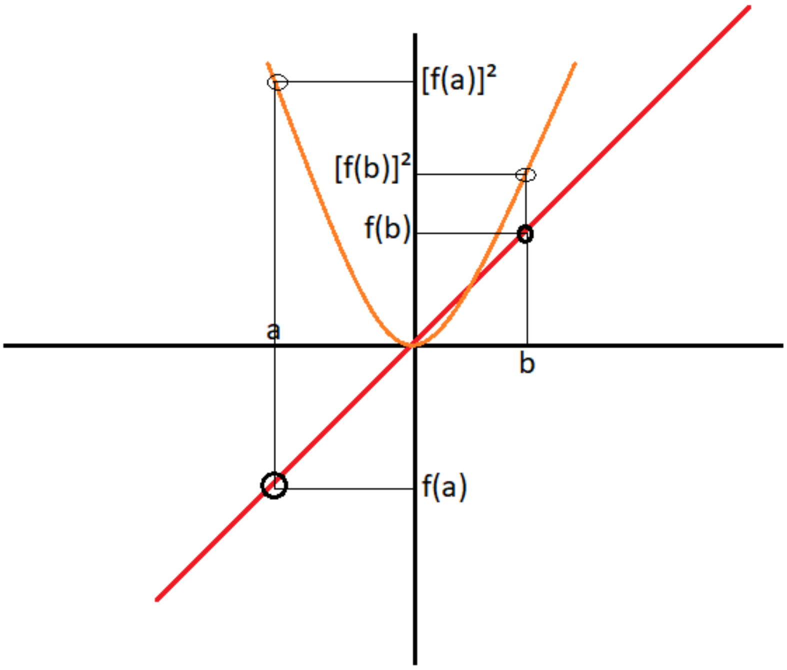
$$\boldsymbol{MRS = -\frac{dx_2}{dx_1}|U_{cnt} = \frac{MU_1}{MU_2} = \frac{\left(\frac{\partial U}{\partial x_1}\right)}{\left(\frac{\partial U}{\partial x_2}\right)}}$$

MRS:

- Graphical interpretation: the slope of the indifference at a certain point.
- Economic interpretation: how many units of x_2 are you willing to give up in order to obtain 1 unit of x_1.

Questions

Question 1



For a linear function $f(x)$, with $a < 0$, $b > 0$, and $|a| > |b|$, we have:

$$f(a) < f(b)$$

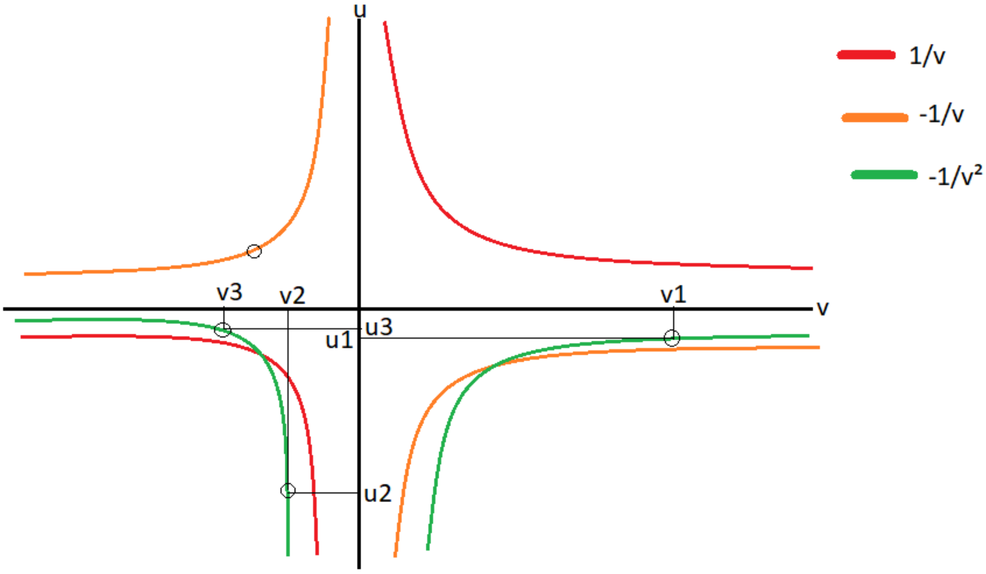
$$[f(a)]^2 > [f(b)]^2$$

This is not a monotonic transformation.

Question 2

(1) $u = 2v - 13$: Is a monotonic transformation.

(2) $u = -\frac{1}{v^2}$



$$v_3 < v_2 < v_1$$

For the function to be monotonic, it would have to have one of the two following alternatives:

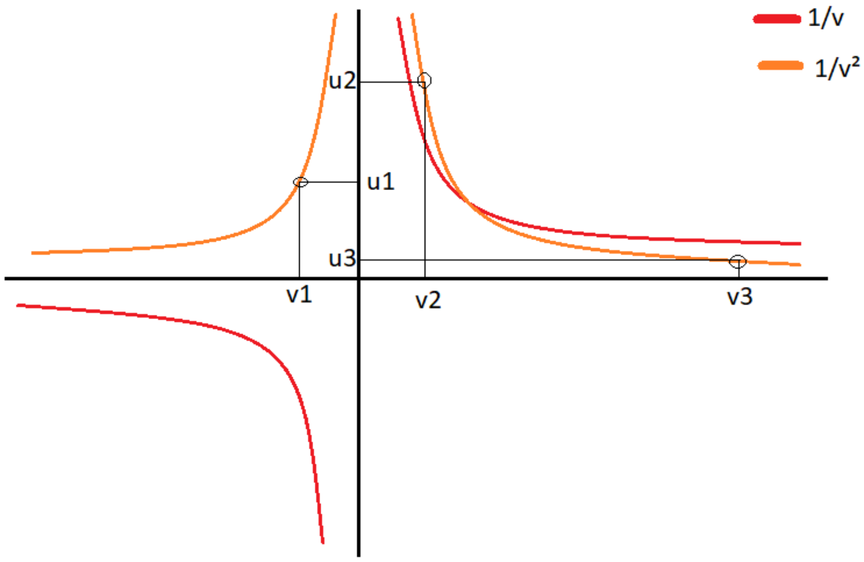
1. $u_3 < u_2 < u_1$
2. $u_3 > u_2 > u_1$

But, we see that:

$$u_2 < u_3 < u_1$$

Then the function is not monotonic.

(3) $u = \frac{1}{v^2}$



$$v_1 < v_2 < v_3$$

For the function to be monotonic, it would have to have one of the two following alternatives:

1. $u_3 < u_2 < u_1$
2. $u_3 > u_2 > u_1$

But, we see that:

$$u_3 < u_1 < u_2$$

Then the function is not monotonic.

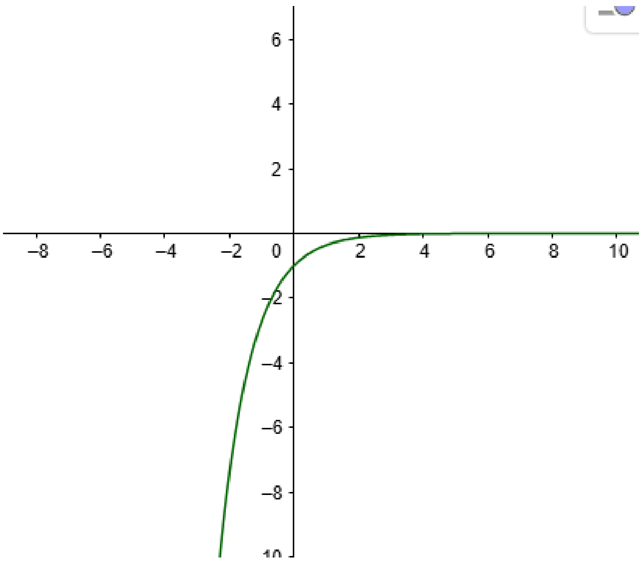
(4) $u = \ln v$: is monotonic.

(5) $u = -e^{-v}$: is monotonic.

There is three operations:

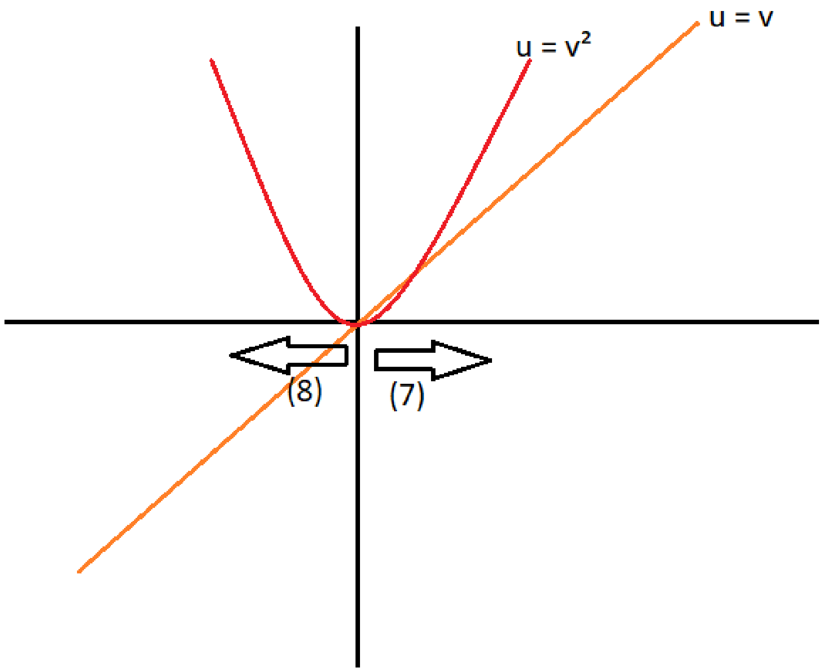
1. $-v$
2. e^{-v}
3. $-e^{-v}$

And of these steps are monotonic transformations.

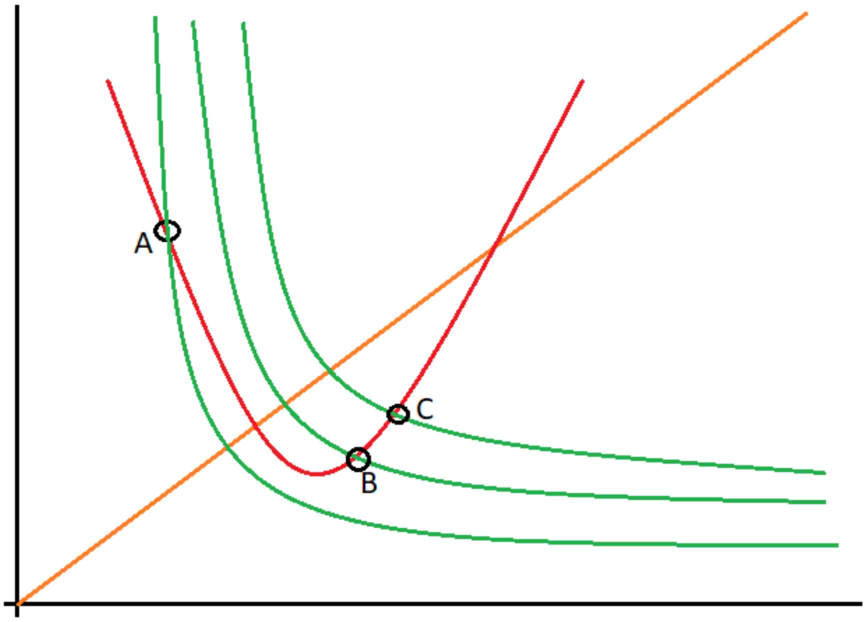


(7) and (8)

1º)



Question 3



Suppose that the red curve is originated from a monotonic transformation of the utility function that generates the green indifference curves. See that, for the green preference, $C \succ B \succ A$, but, for the red preference, $C \sim B \sim A$. Then the transformation is not monotonic, because it doesn't keep the order of the preferences.

Question 4

1º)

$$u(x_1, x_2) = \sqrt{\underbrace{x_1 + x_2}_{\geq 0}}$$

Given that $u(x_1, x_2) \geq 0$, then u^2 is a monotonic transformation (as seen in question 2.7).

$$v(x_1, x_2) = [u(x_1, x_2)]^2 = \left(\sqrt{x_1 + x_2}\right)^2$$

$$\boldsymbol{v(x_1, x_2) = x_1 + x_2 \rightarrow Perfect\ substitutes}$$

2º)

$$u(x_1, x_2) = \frac{v(x_1, x_2)}{13} = \frac{(13x_1 + 13x_2)}{13}$$

$$\boldsymbol{u(x_1, x_2) = x_1 + x_2 \rightarrow Perfect\ substitutes}$$

Question 5

1º)

$$u(x_1, x_2) = x_1 + \underbrace{\sqrt{x_2}}_{f(x_2)}$$

- $f(x_2) = \sqrt{x_2}$

$$\boldsymbol{u(x_1, x_2) = x_1 + f(x_2) \rightarrow Quasilinear\ utility\ function}$$

2º) Calculating MRS_u:

$$u(x_1, x_2) = x_1 + \sqrt{x_2}$$

$$\boldsymbol{\frac{\partial u}{\partial x_1} = 1}$$

$$\frac{\partial u}{\partial x_2} = \left(\frac{1}{2}\right) x_2^{\left(\frac{1}{2}\right)-1} = \left(\frac{1}{2}\right) x_2^{-\left(\frac{1}{2}\right)} = \left(\frac{1}{2}\right)\left(\frac{1}{x_2^{\left(\frac{1}{2}\right)}}\right)$$

$$\boldsymbol{\frac{\partial u}{\partial x_2} = \frac{1}{2\sqrt{x_2}}}$$

$$MRS_u = \frac{\left(\frac{\partial u}{\partial x_1}\right)}{\left(\frac{\partial u}{\partial x_2}\right)} = \frac{1}{\left(\frac{1}{2\sqrt{x_2}}\right)} = \frac{1}{1}\left(\frac{2\sqrt{x_2}}{1}\right)$$

$$\boldsymbol{MRS_u = 2\sqrt{x_2}}$$

2º) Calculating MRS_v:

$$v(x_1, x_2) = x_1^2 + 2x_1\sqrt{x_2} + x_2$$

$$\frac{\partial v}{\partial x_1} = 2x_1 + 2\sqrt{x_2}$$

$$\frac{\partial v}{\partial x_2} = \frac{2x_1}{2\sqrt{x_2}} + 1$$

$$\frac{\partial v}{\partial x_2} = \frac{x_1}{\sqrt{x_2}} + 1$$

$$MRS_v = \frac{\left(\frac{\partial v}{\partial x_1}\right)}{\left(\frac{\partial v}{\partial x_2}\right)} = \frac{(2x_1 + 2\sqrt{x_2})}{\left(\frac{x_1}{\sqrt{x_2}} + \frac{1}{1}\right)} = \frac{2(x_1 + \sqrt{x_2})}{\left(\frac{x_1 + \sqrt{x_2}}{\sqrt{x_2}}\right)} = 2(x_1 + \sqrt{x_2})\left(\frac{\sqrt{x_2}}{x_1 + \sqrt{x_2}}\right)$$

$$= 2\sqrt{x_2}\left(\frac{x_1 + \sqrt{x_2}}{x_1 + \sqrt{x_2}}\right)$$

$$MRS_v = 2\sqrt{x_2}$$

Given that $MRS_u = MRS_v$, then v is a monotonic transformation of u.

Question 6

$$u(x_1, x_2) = \sqrt{x_1 x_2} = (x_1 x_2)^{\frac{1}{2}}$$

$$u(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

$$v(x_1, x_2) = [u(x_1, x_2)]^4 = \left(x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}\right)^4$$

$$v(x_1, x_2) = x_1^2 x_2^2$$