Review
IB Math AI SL

Topic 1: Number and Algebra (Part 2)
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Sequences and Series

Sequence: numbers set by rule
Series: sum of (finite/infinite) terms of sequence

Arithmetic Sequence

Geometric Sequence

Arithmetic Series

Geometric Series

Arithmetic Sequence

Each term *differs* by same fixed number with common difference (**d**)

$$u_{n+1} - u_n = d$$

$$u_n = u_1 + (n-1)d$$

Consider the sequence 8, 13, 18, 23, ...

(a) Show that the sequence is arithmetic.

(b) Find the formula for its general term

(c) Find the 42nd term

(d) Determine whether each number is a member of the sequence:

(i) 153

(ii) 4067

Practice 1

(a) $13 - 8 = 5$
$18 - 13 = 5$
$23 - 18 = 5$
$28 - 23 = 5$

The difference between successive terms is constant.

$\therefore$ The sequence is arithmetic with $u_1 = 8$ and $d = 5$.

(b) $U_n = U_1 + (n-1)d$

Substitute $U_1 = 8$ and $d = 5$,

$U_n = 8 + (n-1)5$

$U_n = 8 + 5n - 5$

$\therefore U_n = 3 + 5n$

(c) From (b) $U_n = 3 + 5n$,

$U_{42} = 3 + 5(42)$

$\therefore U_{42} = 213$

(d) (i) Let $U_n = 153$ (from (b))

$3 + 5n = 153$ (-3)

$5n = 150$ ($\div 5$)

$\therefore n = 30 \in \mathbb{Z}$

$\therefore$ 153 is the 30^{th} term of this sequence.

(ii) Let $U_n = 4067$

$3 + 5n = 4067$

$5n = 4064$

$n = 812\frac{4}{5} \notin \mathbb{Z}$

$\therefore$ 4067 is not a member of this sequence

Find *k* given the consecutive arithmetic means:

(a) $3, k, 11$

(b) $-2, k+4, k^2+11$

(c) $k-5, 2k, 2k^2$

Practice 2

a) $k - 3 = d$

$11 - k = d$

As $d = d$,

$k - 3 = 11 - k$) +k

$2k - 3 = 11$) +3

$2k = 14$) ÷ 2

$\therefore k = 7$

b) $k + 4 - (-2) = k^2 + 11 - (k + 4)$

$k + 6 = k^2 - k + 7$

$0 = k^2 - 2k + 1$

$\therefore k = 1$ (using G.D.C.)

c) $2k - (k - 5) = 2k^2 - 2k$

$k + 5 = 2k^2 - 2k$

$0 = 2k^2 - 3k - 5$

$\therefore k = -1$ or $k = \frac{5}{2}$ (G.D.C.)

Arithmetic SERIES

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{k=1}^{n} u_k$$

$$S_n = \frac{n}{2}(u_1 + u_n) = \frac{n}{2}(2u_1 + (n-1)d)$$

Practice 1

The first four terms of an arithmetic sequence are 51, 45, 39, 33.

(a) Write down the common difference *d*.

(b) Find the 20th term

(c) Find the sum of the first 20 terms

$\hookrightarrow U_1$

(a) $d = 45 - 51$

$\therefore d = -6$

(b) $U_n = U_1 + (n-1)d$

$U_{20} = U_1 + 19d$

$= 51 + 19 \cdot (-6)$

$\therefore U_{20} = -63$

(c) $S_n = \frac{n}{2}(U_1 + U_n)$

$S_{20} = \frac{20}{2}(U_1 + U_{20})$

$= 10(51 + (-63))$

$\therefore S_{20} = -120$

Practice 2

The first term of a finite arithmetic series is 18 and the sum of the series is -210. The common difference is -3. Suppose there are n terms in the series.

$\to u_1$ $\quad$ $\to S_n = -210$ $\quad$ $\to d$

(a) Show that $\frac{n}{2}(39 - 3n) = -210$

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

$$-210 = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

$$\frac{n}{2}\left(2 \cdot 18 + (n-1)(-3)\right) = -210$$

$$\frac{n}{2}(36 - 3n + 3) = -210$$

$$\therefore \frac{n}{2}(39 - 3n) = -210$$

Q.E.D.

(b) Hence find n.

From a,

$$\frac{n}{2}(39 - 3n) = -210$$

$$n(39 - 3n) = -420$$

$$39n - 3n^2 = -420$$

$$0 = 3n^2 - 39n - 420$$

With G.D.C.,

$n = -7$ or $n = 20$

D.M.E. as $n < 0$

$\therefore n = 20 \quad \{n > 0\}$

Geometric Sequence

Each term is obtained by multiplying same non-zero constant with common ratio (**r**)

$$\frac{u_{n+1}}{u_n} = r$$

$$u_n = u_1 \times r^{n-1}$$

Practice 1

Find the general term u_n of the geometric sequence which has:

(a) $u_5 = 324, u_{10} = 78732$

(b) $u_8 = -10, u_{12} = -160$

(a)

$$u_5 = u_1 \cdot r^4 = 324 \quad \cdots (1)$$
$$u_{10} = u_1 \cdot r^9 = 78732 \quad \cdots (2)$$

Now $\dfrac{u_{10}}{u_5} = \dfrac{u_1 \cdot r^9}{u_1 \cdot r^4} = \dfrac{78732}{324}$

$$r^{9-4} = 243$$
$$r^5 = 243$$
$$r = \sqrt[5]{243}$$
$$\therefore r = 3$$

From (1), $u_1 \cdot 3^4 = 324$

$$81 \cdot u_1 = 324$$
$$\therefore u_1 = 4$$

$\therefore$ Thus $u_n = 4 \cdot 3^{n-1}$

(b)

$$u_8 = u_1 \cdot r^7 = -10 \quad \cdots (1)$$
$$u_{12} = u_1 \cdot r^{11} = -160 \quad \cdots (2)$$

$$\frac{u_{12}}{u_8} = \frac{u_1 \cdot r^{11}}{u_1 \cdot r^7} = \frac{-160}{-10}$$

$$r^{11-7} = 16$$
$$r^4 = 16$$
$$r = \pm \sqrt[4]{16}$$
$$r = \pm 2$$

If $r = 2$ & using (1),

$$u_1 \cdot 2^7 = -10$$
$$u_1 \cdot 128 = -10$$
$$u_1 = -\frac{10}{128}$$
$$\therefore u_1 = -\frac{5}{64}$$
$$\therefore u_n = -\frac{5}{64} \cdot 2^{n-1}$$

If $r = -2$ & using (1),

$$u_1 \cdot (-2)^7 = -10$$
$$u_1 \cdot (-128) = -10$$
$$u_1 = \frac{-10}{-128}$$
$$\therefore u_1 = \frac{5}{64}$$
$$\therefore u_n = \frac{5}{64} \cdot (-2)^{n-1}$$

$\therefore u_n = -\frac{5}{64} \cdot 2^{n-1}$ or $u_n = -\frac{5}{64} \cdot (-2)^{n-1}$

Practice 2

Consider the sequence $2, 2\sqrt{3}, 6, 6\sqrt{3}, \ldots$.

(a) Show that the sequence is geometric

(b) Find the formula for its general term

(c) Find the 10th term

(d) Find the first term which exceeds 1000

(a) $\frac{\cancel{2}\sqrt{3}}{\cancel{2}} = \sqrt{3}$

$\frac{6}{2\sqrt{3}} = \frac{\cancel{6}\sqrt{3}}{\cancel{2}\cdot\cancel{3}} = \sqrt{3}$

$\frac{\cancel{6}\sqrt{3}}{\cancel{6}} = \sqrt{3}$

Consecutive terms have a common ratio of $\sqrt{3}$.

$\therefore$ The sequence is geometric with

$u_1 = 2$ and $r = \sqrt{3}$.

(b) $u_n = u_1 \cdot r^{n-1}$

$\therefore u_n = 2(\sqrt{3})^{n-1}$

(c) $u_{10} = 2\cdot(\sqrt{3})^{10-1}$

$= 2(\sqrt{3})^{9}$

$= 2\cdot(\sqrt{3})^{1}\cdot(\sqrt{3})^{8}$

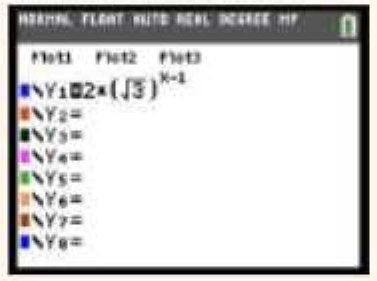
$= 2\sqrt{3}\cdot 3^{\frac{8}{2}}$

$= 2\sqrt{3}\cdot 3^{4}$

$\therefore u_{10} = 162\sqrt{3}$

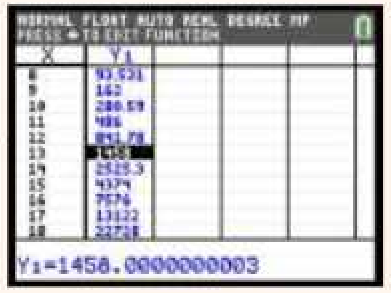
(d) $u_n = 2(\sqrt{3})^{n-1} > 1000$

With G.D.C.,



table

2nd (graph)



$\therefore$ The first term to exceed 1000 is

$u_{13} = 1458$

Geometric SERIES

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{k=1}^{n} u_k$$

$$S_n = \frac{u_1(r^n - 1)}{r - 1}, \langle r \neq 1 \rangle$$

Practice 1

Find the sum of:

(a) $10 + 5 + 2\frac{1}{2} + 1\frac{1}{4} + \ldots$ to 8 terms

(b) $2 + 10 + 50 + 250 + \ldots$ to 10 terms

(c) $\sum_{k=1}^{20} 3 \times (-2)^{k+2}$

(a) $u_1 = 10$

$r = \frac{1}{2}$

$n = 8$

Now $S_n = \frac{u_1 \cdot (r^n - 1)}{r - 1}$

$$S_8 = \frac{10\left(\left(\frac{1}{2}\right)^8 - 1\right)}{\frac{1}{2} - 1}$$

$\therefore S_8 = 19.921875$

(b) $u_1 = 2$, $r = 5$, $n = 10$.

Now $S_n = \frac{u_1 \cdot (r^n - 1)}{r - 1}$

$$S_{10} = \frac{2 \cdot (5^{10} - 1)}{5 - 1}$$

$\therefore S_{10} = 4\ 882\ 812$

(c) $\sum_{k=1}^{20} 3 \cdot (-2)^{k+2} = 3 \cdot (-2)^3 + 3 \cdot (-2)^4 + \dots + 3(-2)^{22}$

$u_1 = 3 \cdot (-2)^3 = -24$

$r = -2$

$n = 20$.

Now $S_n = \frac{u_1 \cdot (r^n - 1)}{r - 1}$

$$S_{20} = \frac{-24\left((-2)^{20} - 1\right)}{-2 - 1}$$

$\therefore S_{20} = 8388600$

See you on next video!

For private tutoring inquiries, please reach me on
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